A semidefinite programming approach to cross-intersection problems

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based on joint work with Sho Suda and Hajime Tanaka

www.cc.u-ryukyu.ac.jp/~hide/slides.pdf

1.
Prototype

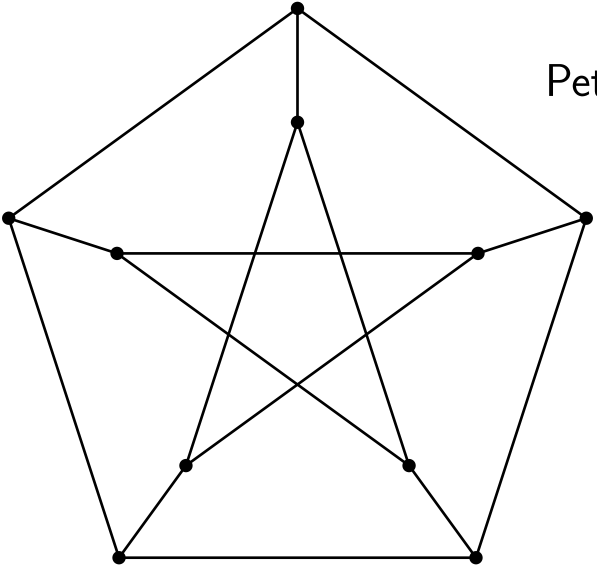
A linear programming approach to an intersection problem

Let $G = (V, E)$ be a graph.

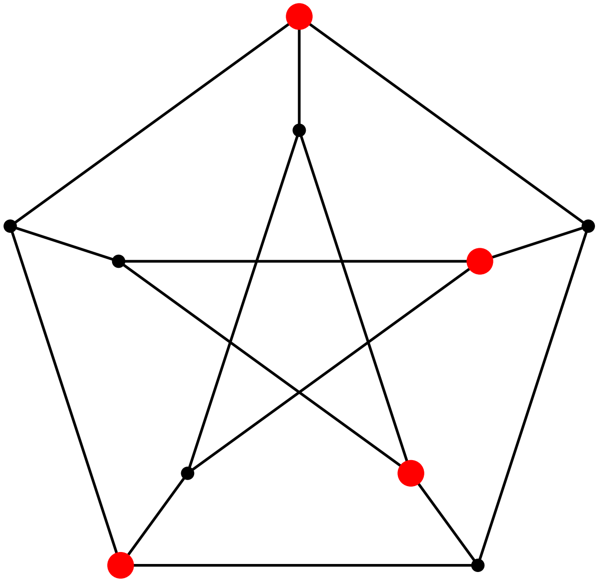
- $U \subset V$ is independent if $uv \notin E$ for all $u, v \in U$.
- The independence number $\alpha(G)$ is the max size of independent sets.

Problem

Determine or estimate the independence number of a given graph.



Petersen graph



$\alpha(G) \geqslant 4$

- Let $A = (a_{x,y})$ be the adjacency matrix of $G = (V, E)$, i.e., a $|V| \times |V|$ matrix such that

$$a_{x,y} = \begin{cases} 1 & \text{if } xy \in E, \\ 0 & \text{otherwise.} \end{cases}$$

- If G is 'symmetric' then eigenvalues of A provide a good bound for $\alpha(G)$:

$$\alpha(G) \leqslant \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} |V|.$$

Hoffman's ratio bound, Delsarte's LP bound, Lovász' θ bound

Example (Petersen graph)

We have

$$\det(xI - A) = (x-3)(x-1)^5(x+2)^4,$$

$$\lambda_{\max} = 3, \quad \lambda_{\min} = -2,$$

and

$$\alpha(G) \leqslant \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} |V| = \frac{-(-2)}{3-(-2)} 10 = 4.$$

- Consequently the independence number of Petersen graph is 4.
- This can be viewed as a result concerning an intersection problem.

- $[n] := \{1, 2, \ldots, n\}$
- $2^{[n]}$: the power set
- $\binom{[n]}{k}$: the set of k-element subsets
- $\mathcal{F} \subset 2^{[n]}$ is intersecting if $F \cap F' \neq \emptyset$ for any $F, F' \in \mathcal{F}$.

Problem

What is the max size of int. families in $\binom{[n]}{k}$?

- $\binom{[n]}{k}$: the set of k-element subsets
- $\mathcal{F} \subset 2^{[n]}$ is intersecting if $F \cap F' \neq \emptyset$ for any $F, F' \in \mathcal{F}$.
- Let

$$\mathcal{H} = \{H \in \binom{[n]}{k} : 1 \in H\}.$$

Then $\mathcal{H}$ is intersecting, and $|\mathcal{H}| = \binom{n-1}{k-1}$.

Erdős–Ko–Rado Theorem

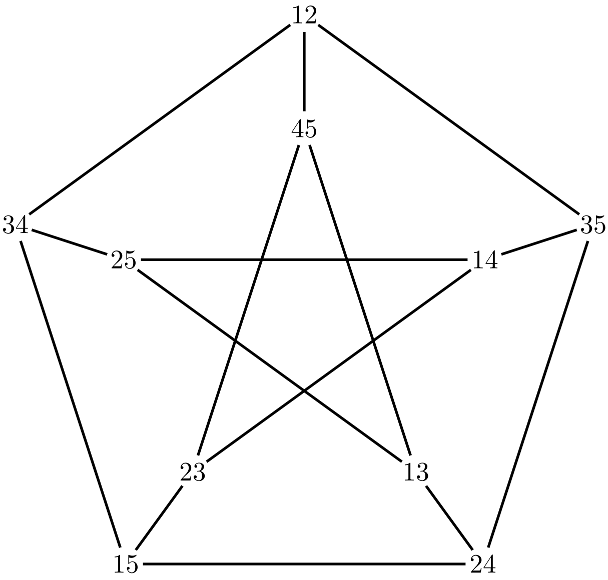
If $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}.$$

- Kneser graph $G_{n,k} = (V, E)$ is defined by

$$V := \binom{[n]}{k}$$
$$E := \{uv : u, v \in V,\ u \cap v = \emptyset\}.$$

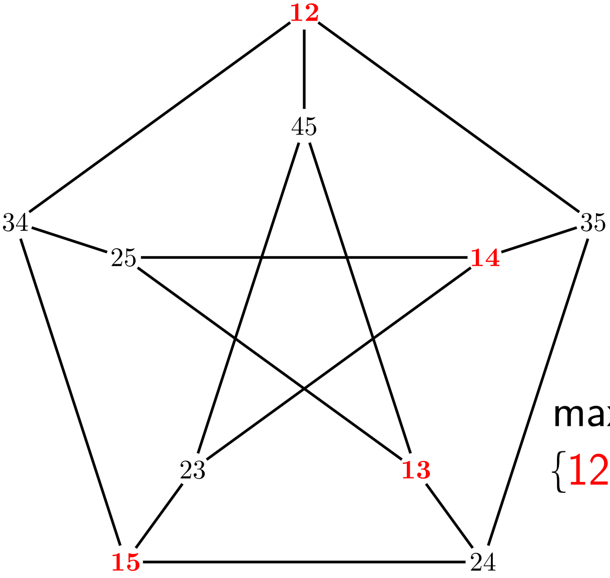
- $U \subset V$ is indep. in $G_{n,k}$ iff U is intersecting.



$G_{5,2}$

$V = \binom{[5]}{2}$

$uv \in E$ iff $u \cap v = \emptyset$.



$V = \binom{[5]}{2}$

$uv \in E$ iff $u \cap v = \emptyset$.

max int. family $\{12, 13, 14, 15\}$

$4 = \binom{5-1}{2-1}$

- One can compute the eigenvalues of the adjacency matrix of the Kneser graph $G_{n,k}$.
- Then the ratio bound gives the upper bound for the independence number of $G_{n,k}$.
- This implies EKR.
- We will extend this idea to solve cross-intersection problems by using SDP.

2.

Cross-intersection problems

and results obtained by SDP approach

Two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross-intersecting if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}, B \in \mathcal{B}$.

Theorem (Pyber, Matsumoto-T, Bey)

If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting, and $n \geqslant 2a$ and $n \geqslant 2b$, then

$$|\mathcal{A}||\mathcal{B}| \leqslant \binom{n-1}{a-1}\binom{n-1}{b-1}.$$

shifting & Kruskal–Katona theorem on shadows,

Katona's cyclic permutation, Sperner inequality

- V : n-dim vector space over $\mathbb{F}_q$
- $\begin{bmatrix} V \\ k \end{bmatrix}$: the set of k-dim subspaces of V
- $\mathcal{A}, \mathcal{B} \subset V$ are cross-intersecting if $A \cap B \neq \{0\}$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem (Suda–Tanaka)

If $\mathcal{A} \subset \begin{bmatrix} V \\ a \end{bmatrix}$ and $\mathcal{B} \subset \begin{bmatrix} V \\ b \end{bmatrix}$ are cross-intersecting, and $n \geqslant 2a$ and $n \geqslant 2b$, then

$$|\mathcal{A}||\mathcal{B}| \leqslant \begin{bmatrix} n-1 \\ a-1 \end{bmatrix} \begin{bmatrix} n-1 \\ b-1 \end{bmatrix}.$$

No combinatorial proof is known so far.

- One can also obtain ‘measure’ variants of cross-intersecting EKR: Suda–Tanaka–T, arXiv:1504.00135
- SDP approach provides a unified way to solve these problems, and it is the only way to prove some of them.

3.
SDP and Weak Duality

Linear Programming problem

$$\text{(primal form)} \quad \begin{array}{ll} \text{minimize} & \mathbf{c}^{\mathsf{T}}\mathbf{x}, \\ \text{subject to} & A\,\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geqslant 0, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ are given, $\mathbf{x} \in \mathbb{R}^n$ is the variable.

$$\text{(dual form)} \quad \begin{array}{ll} \text{maximize} & \mathbf{b}^{\mathsf{T}}\mathbf{y}, \\ \text{subject to} & \mathbf{y}^{\mathsf{T}}A \leqslant \mathbf{c}^{\mathsf{T}}, \end{array}$$

where $\mathbf{y} \in \mathbb{R}^m$ is the variable.

LP problem

(P) min $\mathbf{c}^\top \mathbf{x}$,
subj. to $A\mathbf{x} = \mathbf{b}$,
$\mathbf{x} \geqslant 0$.

(D) max $\mathbf{b}^\top \mathbf{y}$,
subj. to $\mathbf{y}^\top A \leqslant \mathbf{c}^\top$.

Weak duality

If $\mathbf{x}$ is feasible in (P) and $\mathbf{y}$ is feasible in (D), then $\mathbf{c}^\top \mathbf{x} \geqslant (\mathbf{y}^\top A)\mathbf{x} = \mathbf{y}^\top (A\mathbf{x}) = \mathbf{y}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{y}$.

Any feasible solution to (D) gives a lower bound for the objective value of (P).

Semidefinite Programming problem

- An extension of LP.
- Extend the space of the variables from $\mathbb{R}$ to $S\mathbb{R}^{n\times n}$ (the set of $n \times n$ symm. matrices).
- For $A, B \in S\mathbb{R}^{n\times n}$ let

$$A \bullet B := \mathrm{trace}(A^{\mathsf{T}}B) = \sum_{i,j} a_{ij}b_{ij} \in \mathbb{R},$$

- We write $A \succeq 0$ for positive semidefinite A.

SDP problem

$$
\begin{array}{lll}
\text{(P)} & \min & C \bullet X, \\
& \text{subject to} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m, \\
& & X \succeq 0,
\end{array}
$$

$A_i \in S\mathbb{R}^{n \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $C \in S\mathbb{R}^{n \times n}$ are given,
$X \in S\mathbb{R}^{n \times n}$ is the variable.

$$
\begin{array}{lll}
\text{(D)} & \max & \mathbf{b}^{\mathsf{T}} \mathbf{y}, \\
& \text{subject to} & C - \sum_{i=1}^{m} y_i A_i \succeq 0,
\end{array}
$$

$\mathbf{y} \in \mathbb{R}^m$ is the variable.

(Weak duality) $\quad C \bullet X \geqslant \mathbf{b}^{\mathsf{T}} \mathbf{y}.$

4.
SDP for EKR

- Kneser graph $G_{n,k} = (V, E)$ is defined by

$$V := \binom{[n]}{k}$$
$$E := \{uv : u, v \in V,\ u \cap v = \emptyset\}.$$

- $U \subset V$ is indep. in $G_{n,k}$ iff U is intersecting.

Theorem (EKR, reprise)

If $n \geqslant 2k$ then $\alpha(G_{n,k}) = \binom{n-1}{k-1}$.

- Let $G = G_{n,k}$, $N = \binom{n}{k}$, A the adj. matrix.
- Let $U \subset V(G)$ be an independent set.
- Let $\mathbf{x} \in \mathbb{R}^N$ be the char. vector of U and

$$X_U := \frac{1}{|U|}\mathbf{x}\mathbf{x}^\mathsf{T} \in S\mathbb{R}^{N\times N}.$$

- $X_U \succeq 0$, $X_U \geqslant 0$.
- $J \bullet X_U = |U|$.
- $I \bullet X_U = 1$.
- $A \bullet X_U = \frac{1}{|U|}\sum_{i,j} a_{i,j}x_i x_j = 0$.

SDP problem for EKR

$$
\begin{array}{lll}
\text{(P)} & \text{maximize} & J \bullet X, \\
& \text{subject to} & I \bullet X = 1, \quad A \bullet X = 0, \\
& & X \succeq 0, \quad X \geqslant 0.
\end{array}
$$

The max $J \bullet X$ gives an upper bound for the independence number. This is the strengthening of Lovász's θ-function bound due to Schrijver.

SDP problem for EKR

(P) maximize $J \bullet X$,
subject to $I \bullet X = 1, \quad A \bullet X = 0,$
$X \succeq 0, \quad X \geqslant 0.$

(D) minimize α,
subject to $\alpha I - J - \gamma A - Z \succeq 0,$
$Z \geqslant 0.$

Weak Duality

If X is feasible in (P), and (α, γ, Z) is feasible in (D), then $J \bullet X \leqslant \alpha$.

SDP problem for EKR

(D) minimize α,

subject to $\alpha I - J - \gamma A - Z \succeq 0$,

$Z \geqslant 0$.

Optimal feasible solution (unique)

- $\alpha = \binom{n-1}{k-1}$
- $\gamma = \alpha/\lambda_{\min} = -\binom{n-1}{k-1}\binom{n-k-1}{k-1}^{-1}$
- $Z = 0$

This gives us $\alpha(G_{n,k}) \leqslant \binom{n-1}{k-1}$.

5.
SDP for cross-intersecting EKR

Let $G = (V_1 \sqcup V_2, E)$ be a bipartite graph.

- $(U_1, U_2) \subset V_1 \times V_2$ is cross-independent if $uv \notin E$ for all $u \in U_1, v \in U_2$.
- Define $\tilde{\alpha}(G) := \max \sqrt{|U_1||U_2|}$.
- Define $G_{n,k,l} = (V_1 \sqcup V_2, E)$ by

$$V_1 := \binom{[n]}{k}, \quad V_2 := \binom{[n]}{l},$$
$$E := \{uv : (u, v) \in V_1 \times V_2,\ u \cap v = \emptyset\}.$$

- (U_1, U_2) is cross-independent in $G_{n,k,l}$ iff U_1 and U_2 are cross-intersecting.

- Define $G_{n,k,l} = (V_1 \sqcup V_2, E)$ by

$$V_1 := \binom{[n]}{k}, \quad V_2 := \binom{[n]}{l},$$
$$E := \{uv : (u, v) \in V_1 \times V_2,\ u \cap v = \emptyset\}.$$

- $\tilde{\alpha}(G) = \max\{\sqrt{|U_1||U_2|} : (U_1, U_2)_{\text{cross-indep}}\}$.

Theorem (cross-intersecting EKR, reprise)

If $n \geqslant 2k$ and $n \geqslant 2l$, then

$$\tilde{\alpha}(G_{n,k,l}) = \sqrt{\binom{n-1}{k-1}\binom{n-1}{l-1}}.$$

- Let $G = G_{n,k,l}$, $N_1 = \binom{n}{k}$, $N_2 = \binom{n}{l}$.
- Let $A \in \mathbb{R}^{N_1 \times N_2}$ be the bip. adj. matrix.
- Let (U_1, U_2) be cross-independent.
- Let $x_i \in \mathbb{R}^{N_i}$ be the unit char. vector of U_i.
- Let $x = (x_1, x_2) \in \mathbb{R}^{N_1+N_2}$.
- Define

$$X_{U_1,U_2} := x x^{\mathsf{T}} \in S\mathbb{R}^{(N_1+N_2)\times(N_1+N_2)}.$$

- $X_{U_1,U_2} \succeq 0$, $X_{U_1,U_2} \geqslant 0$.

Consider matrices in $S\mathbb{R}^{(N_1+N_2)\times(N_1+N_2)}$

- $\frac{1}{2}\begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix} \bullet X_{u_1,u_2} = \sqrt{|U_1||U_2|}.$
- $\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X_{u_1,u_2} = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix} \bullet X_{u_1,u_2} = 1.$
- $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \bullet X_{u_1,u_2} = 0.$

SDP problem for cross-intersecting EKR

$$\begin{aligned} \text{(P)}\quad & \text{maximize} && \tilde{J} \bullet X, \\ & \text{subject to} && \tilde{I} \bullet X = 2, \quad \tilde{A} \bullet X = 0, \\ & && X \succeq 0, \quad X \geqslant 0, \end{aligned}$$

where

$$\tilde{I} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}, \tilde{J} = \frac{1}{2} \begin{bmatrix} 0 & J \\ J^{\mathsf{T}} & 0 \end{bmatrix}, \tilde{A} = \begin{bmatrix} 0 & A \\ A^{\mathsf{T}} & 0 \end{bmatrix}.$$

This is a bipartite variant of Lovász' θ-bound (and Schrijver's θ'-bound)

SDP problem for cross-intersecting EKR

(D) minimize 2α,

subject to $\alpha\tilde{I} - \tilde{J} - \gamma\tilde{A} - Z \succeq 0, \quad Z \geqslant 0,$

where

$$\tilde{I} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}, \tilde{J} = \frac{1}{2}\begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix}, \tilde{A} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.$$

one of the optimal solutions (1-param. family)

- $2\alpha = \sqrt{\binom{n-1}{k-1}\binom{n-1}{l-1}}, \quad \gamma = -\frac{1}{2}\binom{n-1}{l}\binom{n-k}{l}^{-1}$
- $Z = \epsilon \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad A_1$: adj. mat. of $G_{n,k}$

  $\epsilon = \frac{k-l}{k}\alpha\binom{n-k}{k}^{-1} \quad (k \geqslant l)$

6.

SDP for general cross-intersection problems

SDP for a general cross-int. problem

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- For $i = 1, 2$ let $\mu_i : V_i \to \mathbb{R}_{\geqslant 0}$ be a probability measure, i.e., $\sum_{x \in V_i} \mu(\{x\}) = 1$.
- Let Δ_i be the $|V_i| \times |V_i|$ diagonal matrix with $(\Delta_i)_{xx} := \mu(\{x\})$.
- Let E_{xy} be the $|V_1| \times |V_2|$ matrix with a 1 at (x, y)-entry and 0 elsewhere.

SDP problem for general cross-int. problem

(D) minimize $\alpha + \beta$,

subject to

$$\begin{bmatrix} \alpha\Delta_1 & 0 \\ 0 & \beta\Delta_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & \Delta_1 J \Delta_2 \\ \Delta_2 J^T \Delta_1 & 0 \end{bmatrix} - \sum_{xy \in E} \gamma_{xy} \begin{bmatrix} 0 & E_{xy} \\ E_{yx} & 0 \end{bmatrix} - Z \succeq 0,$$

$$Z \geqslant 0.$$

Theorem

If $U_1 \subset V_1$ and $U_2 \subset V_2$ are cross-independent in G, and $(\alpha, \beta, \gamma_{xy}, Z)$ is feasible in (D), then

$$\mu_1(U_1)\mu_2(U_2) \leqslant (\alpha + \beta)^2.$$

- For $p \in (0,1)$ the product measure $\mu_p : 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ is defined by

$$\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} p^{|F|}(1-p)^{n-|F|}$$

- If $\mathcal{H} = \{H \subset [n] : 1 \in H\}$, then $\mu_p(\mathcal{H}) = p$.

Theorem

If $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross-intersecting, and $p_1 \leqslant 1/2$ and $p_2 \leqslant 1/2$, then

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B}) \leqslant p_1 p_2.$$

Fishburn–Frankl–Freed–Lagarias–Odlyzko (1986) defined a probability measure μ on $2^{[n]}$ with respect to $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (0,1)^n$ by

$$\mu(\mathcal{F}) := \sum_{F\in\mathcal{F}} \prod_{i\in F} p_i \prod_{j\in[n]\setminus F} (1-p_j).$$

Theorem (FFFLO)

Let μ be a measure as above with

- $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n$ and $1/2 \geqslant p_2$.

If $\mathcal{F} \subset 2^{[n]}$ is intersecting, then $\mu(\mathcal{F}) \leqslant p_1$.

Theorem (Suda–Tanaka–T)

Let μ and μ' be FFFLO measures on $2^{[n]}$ with respect to $\mathbf{p}$ and $\mathbf{p}'$, respectively. Suppose that $\mathcal{A}, \mathcal{B} \subset 2^n$ are cross-intersecting.

(i) If $p_1 = \max\{p_i\}$, $1/2 \geqslant \max\{p_2, \ldots, p_n\}$, and $p'_1 = \max\{p'_i\}$, $1/2 \geqslant \max\{p'_2, \ldots, p'_n\}$. Then $\mu(\mathcal{A})\mu'(\mathcal{B}) \leqslant p_1 p'_1$.

(ii) If $p_1 p'_1 = \max\{p_i\, p'_i : 1 \leqslant i \leqslant n\}$, and $p_i, p'_i \leqslant 1/3$ for all i, then $\mu(\mathcal{A})\mu'(\mathcal{B}) \leqslant p_1 p'_1$.
1/3 can be replaced with 1/2 ???

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