The maximum product of measures of cross t-intersecting families

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based on joint work with S.J. Lee and M. Siggers

Notation and Definition

- $[n] := \{1, 2, \ldots, n\}$: the vertex set.
- $\Omega_n := 2^{[n]}$: the power set of $[n]$.
- A family of subsets $\mathcal{F} \subset \Omega_n$.
- $\mathcal{F}$ is t-intersecting if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$.
- For $p \in (0, 1)$ and $\mathcal{F} \subset \Omega_n$ the measure $\mu_p(\mathcal{F})$ is defined by

$$\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} p^{|F|}(1-p)^{n-|F|}.$$

- We write Ω and μ instead of Ω_n and μ_p.

$$\Omega = 2^{[n]}.$$

$$\begin{aligned}
\mu(\Omega) &= \sum_{F \in \Omega} p^{|F|}(1-p)^{n-|F|} \\
&= \sum_{k=0}^{n} \sum_{F \in \binom{[n]}{k}} p^{|F|}(1-p)^{n-|F|} \\
&= \sum_{k=0}^{n} \binom{n}{k} p^{k}(1-p)^{n-k} \\
&= (p + (1-p))^{n} \\
&= 1.
\end{aligned}$$

- Let $\mathcal{F} := \{F \in \Omega : [t] \subset F\}$.
- $\mathcal{F}$ is a t-intersecting family.
- $[n] = [t] \sqcup [t+1, n]$, $\Omega' := 2^{[t+1,n]}$.
- $\mathcal{F} = \{[t] \sqcup G : G \in \Omega'\}$.
- $$\begin{aligned}\mu(\mathcal{F}) &= \sum_{G \in \Omega'} p^t \cdot p^{|G|}(1-p)^{(n-t)-|G|} \\ &= p^t \cdot \mu(\Omega') = p^t\end{aligned}$$
- $\mathcal{F}$ is a t-intersecting family with $\mu(\mathcal{F}) = p^t$.

Theorem

If $p \leq \frac{1}{t+1}$ and $\mathcal{F} \subset \Omega$ is t-intersecting, then $\mu(\mathcal{F}) \leq p^t$.

- Define $\mathcal{F}_r := \{F \subset [n] : |F \cap [t+2r]| \geq t+r\}$.
- $\mathcal{F}_r$ is a t-intersecting family.

Theorem (Ahlswede–Khachatrian)

If $\mathcal{A} \subset \Omega$ is t-intersecting, then

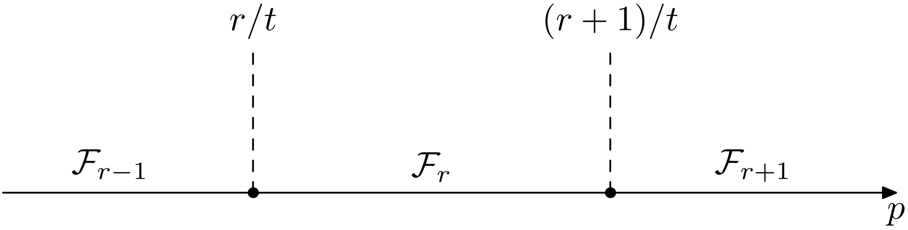
$$\mu(\mathcal{A}) \leq \max_r \mu(\mathcal{F}_r).$$

Moreover equality holds only if $\mathcal{A} \cong \mathcal{F}_r$ for some r.

- If $p \in \left[\frac{r}{t+2r-1}, \frac{r+1}{t+2r+1}\right] := I_{r,t}$, then

$$\mu(\mathcal{F}_0) < \mu(\mathcal{F}_1) < \cdots < \mu(\mathcal{F}_{r-1}) \leq \mu(\mathcal{F}_r)$$
$$\mu(\mathcal{F}_r) \geq \mu(\mathcal{F}_{r+1}) > \mu(\mathcal{F}_{r+2}) > \cdots$$

t-intersecting families with the maximal measures



- Consider the case $t \gg r$.
- $I_{r,t} = \left[\frac{r}{t+2r-1}, \frac{r+1}{t+2r+1}\right] \approx \left[\frac{r}{t}, \frac{r+1}{t}\right]$

- Define $\mathcal{F}_r := \{F \subset [n] : |F \cap [t+2r]| \geq t+r\}$.

Theorem (Ahlswede–Khachatrian, reprise)

If $p \in I_{r,t}$ and $\mathcal{A} \subset \Omega$ is t-intersecting, then

$$\mu(\mathcal{A}) \leq \mu(\mathcal{F}_r).$$

Moreover equality holds only if $\mathcal{A} \cong \mathcal{F}_{r-1}, \mathcal{F}_r, \mathcal{F}_{r+1}$.

- Frankl–Füredi 1991 (uniform version, for $t \gg r$)
- Ahlswede–Khachatrian 1998, Bey–Engel 1998 (method of generating sets)
- Dinur–Safra 2005, T 2005 (from complete EKR)
- Friedgut 2008 (spectral method, for $r = 0$, stability)

- Two families $\mathcal{A}, \mathcal{B} \subset \Omega$ are cross t-intersecting if $|A \cap B| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$.

Question

Is it true that if $p \in I_{r,t}$, and $\mathcal{A}, \mathcal{B} \subset \Omega$ are cross t-intersecting, then

$$\mu(\mathcal{A})\mu(\mathcal{B}) \leq \mu(\mathcal{F}_r)^2\,?$$

Our result: This is true if $t \gg r$.

Theorem 1

For every $r \geq 0$, there is $t_0 = t_0(r)$ such that for all $t \geq t_0$ and $p \in I_{r,t}$ the following holds:
If $\mathcal{A}, \mathcal{B} \subset \Omega$ are cross t-intersecting, then

$$\mu(\mathcal{A})\mu(\mathcal{B}) \leq \mu(\mathcal{F}_r)^2.$$

Equality holds iff

1. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_{r-1}$ and $p = \min I_{r,t}$,
2. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_r$ and $p \in I_{r,t}$,
3. $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_{r+1}$ and $p = \max I_{r,t}$.

Theorem 1 (reprise)

If $t \gg r$, $p \in I_{r,t}$, and $\mathcal{A}, \mathcal{B}$ are cross t-intersecting, then $\mu(\mathcal{A})\mu(\mathcal{B}) \le \mu(\mathcal{F}_r)^2$.

Question about stability

Is it true that if $\mu(\mathcal{A})\mu(\mathcal{B})$ is close to $\mu(\mathcal{F}_r)^2$, then both $\mathcal{A}$ and $\mathcal{B}$ are close to one of $\mathcal{F}_{r-1}, \mathcal{F}_r, \mathcal{F}_{r+1}$?

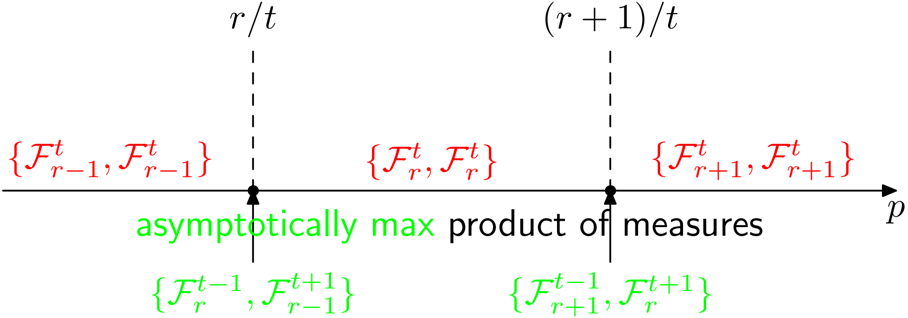
- Assume further that $\mathcal{A}$ and $\mathcal{B}$ are shifted.
  if $A \in \mathcal{A}$, $A \cap \{i,j\} = \{j\}$, $i < j$, then $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}$.
- Then it is true if $p \in (\frac{r+\epsilon}{t}, \frac{r+1-\epsilon}{t})$ and $t > t_0(r,\epsilon)$.
- If $p \approx \frac{r}{t}$ or $\frac{r+1}{t}$, then there are other structures!

Almost optimal cross t-intersecting families at $p = \frac{r}{t}$

- Write $\mathcal{F}_r^t$ instead of $\mathcal{F}_r$.
- $\mathcal{F}_r^t := \{F \in \Omega : |F \cap [t+2r]| \geq t + r\}$.
- $\mathcal{F}_r^{t-1} = \{F \in \Omega : |F \cap [t+2r-1]| \geq t + r - 1\}$.
- $\mathcal{F}_{r-1}^{t+1} = \{F \in \Omega : |F \cap [t+2r-1]| \geq t + r\}$.
- $\mathcal{G} := \mathcal{F}_r^{t-1}$ is $(t-1)$-intersecting.
- $\mathcal{H} := \mathcal{F}_{r-1}^{t+1}$ is $(t+1)$-intersecting.
- $\mathcal{H} \subset \mathcal{F}_r^t \subset \mathcal{G}$.
- $\mathcal{G}$ and $\mathcal{H}$ are cross t-intersecting.
- If $p = \frac{r}{t}$ then $\mu(\mathcal{F}_r^t)^2 > \mu(\mathcal{G})\mu(\mathcal{H})$ but

$$\lim_{t\to\infty} \frac{\mu(\mathcal{F}_r^t)^2}{\mu(\mathcal{G})\mu(\mathcal{H})} = 1.$$

cross t-int. families with max product of measures



What happens at around $p = r/t$?

- Let $\mathcal{G} := \mathcal{F}_r^{t-1}$, $\mathcal{H} := \mathcal{F}_{r-1}^{t+1}$.
- if $p = r/t$ then
$$\mu(\mathcal{F}_r^t)^2 \approx \mu(\mathcal{G})\mu(\mathcal{H}) \approx \mu(\mathcal{F}_{r-1}^t)^2.$$
- if $p = (r-\epsilon)/t$ then
$$\mu(\mathcal{F}_r^t)^2 < \mu(\mathcal{G})\mu(\mathcal{H}) < \mu(\mathcal{F}_{r-1}^t)^2.$$
- if $p = (r+\epsilon)/t$ then
$$\mu(\mathcal{F}_r^t)^2 > \mu(\mathcal{G})\mu(\mathcal{H}) > \mu(\mathcal{F}_{r-1}^t)^2.$$

Theorem 2

For every $r \geq 1$ and every $0 < \epsilon < 1/2$, there is $t_0 = t_0(r, \epsilon)$ such that for all $t \geq t_0$ and

$$p \in \left(\frac{r+\epsilon}{t}, \frac{r+1-\epsilon}{t} \right) \subsetneq I_{r,t}$$

the following holds: If $\mathcal{A}, \mathcal{B} \subset \Omega$ are shifted cross t-intersecting families, and

$$\sqrt{\mu(\mathcal{A})\mu(\mathcal{B})} = (1-\eta)\,\mu(\mathcal{F}_r^t),$$

for some $0 < \eta < \dfrac{\epsilon}{2(r+1)}$, then

$$\mu(\mathcal{A} \triangle \mathcal{F}_r^t) + \mu(\mathcal{B} \triangle \mathcal{F}_r^t) < 3\eta\,\mu(\mathcal{F}_r^t).$$

Let $m := \frac{\epsilon}{2(r+1)}$, and $\mathcal{F} := \mathcal{F}_r^t$.

Theorem 2 (reprise)

Let $p \in (\frac{r+\epsilon}{t}, \frac{r+1-\epsilon}{t})$, $0 < \eta < m$. If $\mathcal{A}, \mathcal{B}$ are shifted cross t-intersecting and $\sqrt{\mu(\mathcal{A})\mu(\mathcal{B})} = (1-\eta)\,\mu(\mathcal{F})$, then $\mu(\mathcal{A} \triangle \mathcal{F}) + \mu(\mathcal{B} \triangle \mathcal{F}) < 3\eta\,\mu(\mathcal{F})$.

m cannot be replaced with $\gamma := 1 - \sqrt{1 - \frac{\epsilon}{r+1}} > m$.

Example

Let $p = \frac{r+1-\epsilon}{t}$, $\mathcal{G} = \mathcal{F}_{r+1}^{t-1}$, $\mathcal{H} = \mathcal{F}_r^{t+1}$, and $\mathcal{F} := \mathcal{F}_r^t$.
Then $\sqrt{\mu(\mathcal{G})\mu(\mathcal{H})} \approx (1-\gamma)\,\mu(\mathcal{F})$, but
$\mu(\mathcal{G} \triangle \mathcal{F}) = \mu(\mathcal{G} \setminus \mathcal{F}) \gg \mu(\mathcal{F})$.

Some problems

Theorem 1

For every $r \geq 0$, there is $t_0 = t_0(r)$ such that for all $t \geq t_0$ and $p \in I_{r,t}$ the following hold: If $\mathcal{A}, \mathcal{B} \subset \Omega$ are cross t-intersecting, then $\mu(\mathcal{A})\mu(\mathcal{B}) \leq \mu(\mathcal{F}_r)^2$.

Problem 1

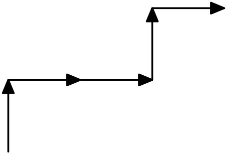
Does this true for all $t \geq 1$?

Problem 2

In Theorem 2 we assumed that both families are shifted. Does the same hold without assuming shiftedness?

Some ideas for proof (based on Frankl–Füredi 1991)

- Regard a subset $F \subset [n]$ as a walk defined below.



- Example: $F = \{1, 4\} \subset [5]$
- For $\mathcal{F} \subset \Omega$ let

$$\lambda(\mathcal{F}) := \max\{l : \text{if } F \in \mathcal{F} \text{ then } F \text{ hits } y = x + l\}.$$

- $\mu(\mathcal{F}) \leq \alpha^{\lambda(\mathcal{F})}$, where $\alpha := \dfrac{p}{1-p}$.
- If $\mathcal{F}$ is shifted and t-intersecting, then $\lambda(\mathcal{F}) \geq t$, and $\mu(\mathcal{F}) \leq \alpha^t$.

- If $\mathcal{A}, \mathcal{B}$ are shifted and cross t-intersecting, then $\lambda(\mathcal{A}) + \lambda(\mathcal{B}) \geq 2t$. Thus

$$\mu(\mathcal{A})\mu(\mathcal{B}) \leq \alpha^{\lambda(\mathcal{A})}\alpha^{\lambda(\mathcal{B})} = \alpha^{\lambda(\mathcal{A})+\lambda(\mathcal{B})} \leq \alpha^{2t}.$$

- Let $u = \lambda(\mathcal{A})$ and divide $\mathcal{A} = \tilde{\mathcal{A}} \sqcup \mathcal{A}'$, where

$$\tilde{\mathcal{A}} = \{A \in \mathcal{A} : A \text{ hits } y = x + (u+1)\},$$
$$\mathcal{A}' = \mathcal{A} \setminus \tilde{\mathcal{A}}.$$

- $\mu(\tilde{\mathcal{A}})$ is small, and $\mathcal{A}'$ is essential.

- Let $\mathcal{A}, \mathcal{B}$ be shifted and cross t-intersecting families.
- Let $u = \lambda(\mathcal{A})$, $v = \lambda(\mathcal{B})$.
- Then $u + v \geq 2t$ and $\mu(\mathcal{A})\mu(\mathcal{B}) \leq \alpha^{u+v}$.
- The case $u + v = 2t$ is essential.
- There are unique $s, s' \geq 0$ such that

$$\mathcal{A}' \subset \mathcal{F}_s^u, \quad \mathcal{B}' \subset \mathcal{F}_{s'}^v.$$

- $u = t - |s - s'|$, $v = t + |s - s'|$.

$$\mu(\mathcal{A})\mu(\mathcal{B}) \approx \mu(\mathcal{A}')\mu(\mathcal{B}') \\ \leq \mu(\mathcal{F}_s^u)\mu(\mathcal{F}_{s'}^v) \leq \mu(\mathcal{F}_r^t)^2.$$