Recent applications of the slice rank method

Norihide Tokushige

Ryukyu University

RIMS 共同研究「代数的組合せ論と関連する群と代数の研究」
@RIMS Kyoto, Dec 16th, 2019

This talk is based on joint work with Masato Mimura.

Large subset without 3-AP and the Slice Rank

- Let G be a finite abelian group.
- A 3-AP (arithmetic progression of length 3) in G is a triple of elements $\{x, x+d, x+2d\}$ with $d \neq 0$.

Problem

Suppose that $X \subset G$ contains no 3-AP.
Then how large $|X|$ can be?

For example,

- $G = \mathbb{Z}_n$ ... Roth's theorem, $|X| \ll n/(\log\log n)$.
- $G = \mathbb{Z}_4^n$ ... Croot–Lev–Pach (2016), $|X| < 3.61^n$.
- $G = \mathbb{F}_3^n$ ... today's topic, $|X| < 2.76^n$.

Let

$$m(n) := \max\{|X| : X \subset \mathbb{F}_3^n \text{ has no 3-AP}\}.$$

Trivially $m(n) < 3^n$.

Theorem (Edel 2004, Ellenberg–Gijswijt 2016)

$$2.21^n < m(n) < 2.76^n.$$

- lower bound : by construction, e.g., $\{0, 1\}^n$.
- upper bound : by polynomial method due to CLP.
- Tao reformulated the proof in terms of “slice rank.”

Tao's idea: Extend the fact that

"a diagonal matrix without zero diagonal is of full rank."

Setup

- Let X be a finite set, and $\mathbb{F}$ be a field.
- Let $f : X^3 \to \mathbb{F}$.
- f is *sliced* if it is written in one of the following:

$$f(x, y, z) = a(x)b(y, z) \text{ or } a(y)b(x, z) \text{ or } a(z)b(x, y).$$

- Any function $f : X^3 \to \mathbb{F}$ can be written as a sum of $|X|$ sliced functions.
- Define the *slice rank* of f by

$$\mathrm{sr}(f) := \min \left\{ r : f = \sum_{i=1}^{r} g_i, \ \ g_i \text{ is sliced} \right\}$$

- We have $\mathrm{sr}(f) \le |X|$.

Lemma (Tao)

Suppose that $f : X^3 \to \mathbb{F}$ satisfies the diagonal condition

$$f(x, y, z) \neq 0 \text{ if and only if } x = y = z.$$

Then $\mathrm{sr}(f) = |X|$.

How to use the slice rank to bound the size of 3-AP-free subset

Proof of the Theorem (upper bound).
If $X \subset \mathbb{F}_3^n$ has no 3-AP, then $|X| < c^n$ for some $c < 3$.

Observation

If $x, y, z \in \mathbb{F}_3^n$ form a 3-AP (in this order), then

$$x + z = 2y,$$

or equivalently

$$x + y + z = 0.$$

On the other hand if $x, y, z \in \mathbb{F}_3^n$ satisfy

$$x + y + z = 0,$$

then they form a 3-AP, or $x = y = z$.

Proof (continued)

Let $X \subset \mathbb{F}_3^n$ be 3-AP-free. For all $x, y, z \in X$ we have

$$x + y + z = 0 \text{ if and only if } x = y = z.$$

Write $x = (x_1, \ldots, x_n)$, and define $f : X^3 \to \mathbb{F}_3$ by

$$f(x, y, z) = \prod_{i=1}^{n} \left((x_i + y_i + z_i)^2 - 1 \right)$$

Then f fulfills the diagonal condition

$$f(x, y, z) \neq 0 \text{ if and only if } x = y = z.$$

By Tao's lemma we get $\mathrm{sr}(f) = |X|$.

Proof (ending)
Define $g : (0,1) \to \mathbb{R}$ by $g(x) = x^{-2/3}(1 + x + x^2)$.
Then $g(x)$ takes minimum at $\alpha \in (0,1)$, and

$$g(x) \le g(\alpha) < 2.76.$$

A technical lemma

The slice rank of the function

$$f(x,y,z) = \prod_{i=1}^{n} \left((x_i + y_i + z_i)^2 - 1\right)$$

satisfies

$$\mathrm{sr}(f) < g(\alpha)^n < 2.76^n.$$

Summary of the proof

Let $X \subset \mathbb{F}_3^n$ be 3-AP-free. Define $f : X^3 \to \mathbb{F}_3$ by

$$f(x, y, z) = \prod_{i=1}^{n} \left((x_i + y_i + z_i)^2 - 1\right).$$

Then f has the diagonal condition with small slice rank:

- $f(x, y, z) \neq 0$ iff $x = y = z$, and
- $\mathrm{sr}(f) < 2.76^n$.

Thus by Tao's lemma we have

$$|X| = \mathrm{sr}(f) < 2.76^n.$$

The slice rank method

- Suppose that $X \subset G$ has property (P), and we want to say $|X| < m$.
- Find a function $f : X^k \to \mathbb{F}$ which reflects (P) in such a way that for all $x_1, \ldots, x_k \in X$,

$$f(x_1, \ldots, x_k) \neq 0 \text{ iff } x_1 = \cdots = x_k,$$

  and moreover $\mathrm{sr}(f) < m$.
- Then by Tao's lemma we have

$$|X| = \mathrm{sr}(f) < m.$$

Applications

- Let $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k} = \{F \subset [n] : |F| = k\}$.
- Three distinct subsets $A, B, C \subset [n]$ form a *sunflower* if $A \cap B = B \cap C = C \cap A$.
- Identify $A \subset [n]$ with its char. vec. $a = (a_1, \ldots, a_n)$ where $a_i = 1$ if $i \in A$, and $a_i = 0$ if $i \notin A$.

Theorem 2 (Naslund and Sawin 2016)

If $X \subset \binom{[n]}{k}$ contains no sunflowers then $|X| < 1.9^n$.

Define $f : X^3 \to \mathbb{R}$ by $f(x, y, z) = \prod_{i=1}^{n}(x_i + y_i + z_i - 2)$.
Then f satisfies the diag. cond., and $\mathrm{sr}(f) < 1.9^n$.

Let $\chi_k(\mathbb{R}^n)$ denote the minimum number of colors needed to color $\mathbb{R}^n$ so that it does not contain a monochromatic regular k-simplex of side length 1.

- $5 \leq \chi_1(\mathbb{R}^2) \leq 7$.
- $(1.2 + o(1))^n \leq \chi_1(\mathbb{R}^n) \leq (3 + o(1))^n$.
- $(1 + \frac{1}{2^{2^{k+4}}} + o(1))^n \leq \chi_k(\mathbb{R}^n) \leq (1 + (2 + \frac{2}{k})^{\frac{1}{2}} + o(1))^n$.
  F–R, L–R, etc.
- $\chi_2(\mathbb{R}^n) > (1 + c + o(1))^n$.
  $c = 0.00085$. Sagdeev 2018. (26 pages)
  $c = 0.01446$. Naslund 2019^+. (4 pages)
  arXiv:1909.09856 by Slice Rank Method.

The slice rank method is useful.
But there are some difficulties:

- Dealing with a *system of equations* is not easy.
- We usually require that elements of X satisfies the given equation iff all the variables are the same. E.g., for the 3-AP-free case we need

$$x + y + z = 0 \quad \text{iff} \quad x = y = z.$$

In some cases this condition is *too strong*.

What about 4-AP?

$$m_4(n) := \max\{|X| : X \subset \mathbb{F}_p^n \text{ has no } 4\text{-AP}\}.$$

An important open problem

Is it true that $m_4(n) < (cp)^n$ for some $0 < c < 1$?

It follows from the Hales-Jewett Theorem that

$$\lim_{n\to\infty} \frac{m_4(n)}{p^n} = 0.$$

$(x, y, z, w) \in (\mathbb{F}_p^n)^4$ forms a 4-AP
$\iff$ both (x, y, z) and (y, z, w) form 3-APs.

$X \subset \mathbb{F}_p^n$ has no 4-AP $\iff$

if $x, y, z, w \in X$ satisfies

$$\begin{cases} x - 2y + z = 0 \\ y - 2z + w = 0 \end{cases} \quad \text{then} \quad x = y = z = w.$$

One can apply SRM, but the outcome is $|X| \leq p^n$.

In general it is not so easy to get a non-trivial bound for *a system of equations* using SRM.

Suppose that $X \subset \mathbb{F}_p^n$ has no non-trivial solution to

$$x - y + z - w = 0,$$

that is, if $x, y, z, w \in X$ satisfy the equation, then $x = y = z = w$.
In this case one can get $|X| < (cp)^n$ for $c < 1$ by SRM.

But if $a, b \in X$ with $a \neq b$ then

$$a - a + b - b = 0.$$

So X cannot have more than one element!

Theorem

Let $X \subset \mathbb{F}_p^n$. Suppose that the equation

$$x - y + z - w = 0$$

has no solution with 4 *distinct* elements of X. Then,

- $|X| < \sqrt{p}^{\,n} + \frac{1}{2}$.
- If n is even, one can construct X with $|X| = \sqrt{p}^{\,n}$.

Counting a sum set yields the upper bound (Ruzsa's idea for weak Sidon sets).

An upper bound coming from SRM is much worse.

Nevertheless there are some positive results obtained using SRM.

Theorem (Sauermann 2019$^+$, arXiv:1904.09560)

Let $X \subset \mathbb{F}_p^n$. Suppose that the equation

$$x_1 + x_2 + \cdots + x_p = 0$$

has no solution with p distinct elements of X. Then

$$|X| < C_p\,(2\sqrt{p})^n.$$

Proved by SRM plus additional combinatorial ideas.
This gives a good upper bound for Erdős–Ginzburg–Ziv constant.

A sequence of $k-1$ 0's and $k-1$ 1's $(0,0,\dots,0,1,1,\dots,1)$ does not contain a subsequence of length k which sums to $0 \pmod{k}$.

Theorem (Erdős–Ginzburg–Ziv)

Every integer sequence of length $2k-1$ contains a subsequence of length k which sums to $0 \pmod{k}$.

Let $s(\mathbb{F}_p^n)$ denote the minimum number s such that any sequence in $\mathbb{F}_p^n$ of length s contains a subsequence of length p which sums to 0.

Theorem (Sauermann 2019^+)

$s(\mathbb{F}_p^n) < (p-1)\,C_p\,(2\sqrt{p})^n + 1.$

Theorem (arXiv:1909.10509)

Let $X \subset \mathbb{F}_p^n$. Suppose that the system of equations

$$x - y + z - u = 0, \quad x - 2z + v = 0$$

has no solution with 5 distinct elements of X, and $|X|$ is maximum. Then

$$(c_1 p)^n < |X| < (c_2 p)^n$$

for some constants $0 < c_1 < c_2 < 1$.

Probabilistic sampling argument gives a simpler proof with a better upper bound. (Sauermann '19/12/13)

Theorem

Let $X \subset \mathbb{F}_p^n$. Suppose that the system of equations

$$x + y + z - 3u = 0, \quad x + y + v - 3w = 0$$

has no solution with 6 distinct elements of X, and $|X|$ is maximum. Then

$$(c_1 p)^n < |X| < (c_2 p)^n$$

for some constants $0 < c_1 < c_2 < 1$.

maybe probabilistic sampling argument doesn't work for this case?

Probabilistic sampling is also useful.

Theorem

Let $X \subset \mathbb{F}_p^n$. Suppose that the system of equations

$$\begin{cases} x_1 + x_3 = x_2 + x_4, \\ y_1 + y_3 = y_2 + y_4, \\ x_1 + y_2 = x_2 + y_1, \\ x_1 + y_4 = x_4 + y_1. \end{cases}$$

has no solution with 8 distinct elements of X, and $|X|$ is maximum. Then

$$c\,(\sqrt{p})^n < |X| < 4\left(p^{\frac{7}{8}}\right)^n.$$

An example of probabilistic sampling argument.

Let $A \subset \mathbb{F}_p^n$. Suppose that the system of equations

$$x - y + z - u = 0, \quad x - 2z + v = 0$$

has no solution with 5 distinct elements of A. In this case we say that A is W-free for short.

We want to show that if A is W-free then $|A|$ is small.

We know that if A is 3-AP-free then $|A|$ is small.

Suppose that $A \subset \mathbb{F}_p^n$ is W-free. Then it follows that

- A contains no 5-AP,
- A contains no two disjoint 3-AP with the same diff.

This yields that $\#$(3-AP in A) is small.

We want to show that $|A|$ is small. For contradiction assume that $|A|$ is large.

Let $B \subset A$ be a random subset created by selecting each $a \in A$ with some well-chosen probability q.

Let $X = |B|$. Then $\mathbb{E}[X] = q|A|$ is large.

Let $Y = \#$(3-AP in B).
Then $\mathbb{E}[Y] = \#$(3-AP in A) $\cdot q^3$ is small.

Thus $\mathbb{E}[X - Y]$ is large.
Then we find a large 3-AP-free subset. contradiction!