# Alon’s transmitting problem in Hamming graphs

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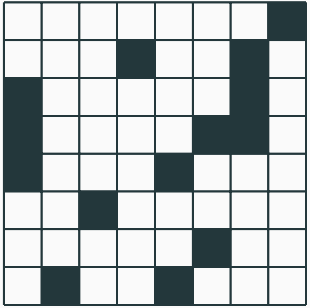
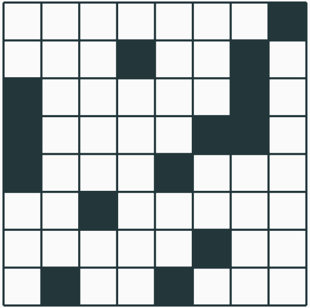
# A lights puzzle



Consider $n \times n$ array of lights.
Each light has two states.
To each row and column, there is a switch.
Turning the switch changes the state of each light in that row and column.
Goal: minimize the discrepancy!



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David Gale (attributed),
Elwyn Berlekamp (1960's tea room of math dept in Bell Labs),
Leo Moser (conjecture),
János Komlós and Miklós Sulyok (sufficiently large $n$, 1970)
József Beck and Joel Spencer (complete solution, 1983)

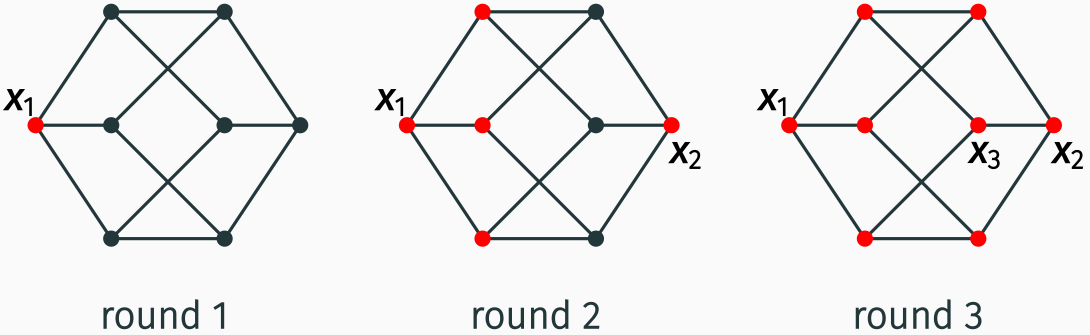
**Lemma (Beck–Spencer 1983)**

For $1 \le i \le n$, let $\boldsymbol{a}_i \in \{\pm 1\}^n$ be given.
Then there is $\boldsymbol{x} \in \{\pm 1\}^n$ such that $|\boldsymbol{a}_i \cdot \boldsymbol{x}| < 2i$ for all $1 \le i \le n$.

- This lemma is a main tool for analyzing the puzzle.
- The lemma is proved by the floating variable method introduced by József Beck and Tibor Fiala (1981).
- We will extend the lemma from $\{\pm 1\}^n$ to $\{1, 2, \ldots, q\}^n$. (multicolor setting)

# Alon's problem

- A finite graph $G$ is given.
- There is a Sender outside the graph, and Sender has a sequence of vertices $x_1, x_2, \ldots$ (called a burning sequence).
- Sender sends a message to a vertex $x_i$ at round $i$.
- A vertex received the message at any round will transmit it to its neighbors at the next round.
- What is the minimum length of a sequence required for all vertices to receive the message?
- Let $b(G)$ be the minimum length (called a burning number).




- $(x_1, x_2, x_3)$ is a burning seq. of length 3, so $b(G) \leq 3$.
- Indeed $b(G) = 3$.

# Some facts about burning number

- $b(G)$ was first introduced by Noga Alon (1992), but recently the concept has been rediscovered and called "burning number."
- $b(K_n) = 2, \ \ b(P_n) = \lceil \sqrt{n} \rceil$. Dense $G$ likely has small $b(G)$.
- Burning number conjecture (Bonato et al. 2016)
  For every connected $n$-vertex graph $G$, $b(G) \leq \lceil \sqrt{n} \rceil$.
- Graph burning problem is NP-complete.
  (Instance) $n$-vertex graph $G$ and $k \geq 2$. (Question) Is $b(G) \leq k$?
- It looks interesting to find a graph $G$ which is not so dense, but $b(G)$ is relatively small. How about hypercube?

Let $G_n$ denote the $n$-dim hypercube, that is,

- $V(G_n) = \{0,1\}^n = \{(v_1, \ldots, v_n) : v_i \in \{0,1\}\}$, and
- two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if $\#\{i : u_i \neq v_i\} = 1$.

**Theorem (Alon 1992)**

$b(G_n) = \lceil \frac{n}{2} \rceil + 1$.

- $b(G_n) \leq \lceil \frac{n}{2} \rceil + 1$ is easy. Let $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{1}$.
- $b(G_n) \geq \lfloor \frac{n}{2} \rfloor + 1$ follows from Beck–Spencer Lemma.
- $b(G_n) \geq \lceil \frac{n}{2} \rceil + 1$ is more difficult. (Alon's trick)

# Extension to Hamming graphs

Let $H = H(n, q)$ denote the Hamming graph, that is,

- $V(H) = [q]^n = \{(v_1, \dots, v_n) : v_i \in \{1, 2, \dots, q\}\}$, and
- two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent if $\#\{i : u_i \neq v_i\} = 1$.
- $n$-dim hypercube is $H(n, 2)$.
- Hamming distance $d(\boldsymbol{u}, \boldsymbol{v}) := \#\{i : u_i \neq v_i\}$.

Let $H = H(n, q)$ denote the Hamming graph, that is,

- $V(H) = [q]^n = \{(v_1, \ldots, v_n) : v_i \in \{1, 2, \ldots, q\}\}$, and
- two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent if $\#\{i : u_i \neq v_i\} = 1$.

**Our result**

$$\lfloor (1 - \tfrac{1}{q})n \rfloor + 1 \leq b(H(n, q)) \leq \lfloor (1 - \tfrac{1}{q})n + \tfrac{q+1}{2} \rfloor.$$

- The upper bound is easy. (by construction)
- For the lower bound we extend Beck–Spencer Lemma.
- If $n \gg q$ then the upper bound is true $b(H)$?

**Lemma (Beck–Spencer 1983)**

For $1 \le i \le n$, let $\mathbf{a}_i \in \{\pm 1\}^n$ be given.
Then there is $\mathbf{x} \in \{\pm 1\}^n$ such that $|\mathbf{a}_i \cdot \mathbf{x}| < 2i$ for all $1 \le i \le n$.

We can think of $\{\pm 1\}^n$ as the vertex set of $H(n,2)$. Then

$$\mathbf{a} \cdot \mathbf{x} = n - 2d(\mathbf{a}, \mathbf{x}).$$

**Lemma (BS restated)**

For $1 \le i \le n$, let $\mathbf{v}_i$ be a given vertex in $H(n,2) \cong G_n$.
Then there is $\mathbf{w}$ such that $|n - 2d(\mathbf{v}_i, \mathbf{w})| < 2i$ for all $1 \le i \le n$.

Thus $d(\mathbf{v}_i, \mathbf{w}) > \frac{n}{2} - i$ and $b(G_n) > \frac{n}{2}$.

**Lemma (Beck–Spencer)**

For $1 \leq i \leq n$, let $\boldsymbol{v}_i$ be a given vertex in $H(n, 2)$.
Then there is $\boldsymbol{w}$ such that $|n - 2d(\boldsymbol{v}_i, \boldsymbol{w})| < 2i$ for all $1 \leq i \leq n$.

**Our lemma (multicolor Beck–Spencer)**

For $1 \leq i \leq n$, let $\boldsymbol{v}_i$ be a given vertex in $H(n, q)$.
Then there is $\boldsymbol{w}$ such that $|(1 - \frac{1}{q})n - d(\boldsymbol{v}_i, \boldsymbol{w})| < i$ for all $1 \leq i \leq n$.

Thus $d(\boldsymbol{v}_i, \boldsymbol{w}) > (1 - \frac{1}{q})n - i$ and $b(H(n, q)) > (1 - \frac{1}{q})n$.

# Proof ideas of Beck–Spencer

**Beck–Spencer Lemma (the case $n = 4$)**

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \{\pm 1\}^4$ be given. Then there is $\boldsymbol{x} \in \{\pm 1\}^4$ such that $|\boldsymbol{a}_1 \cdot \boldsymbol{x}| < 2$, $|\boldsymbol{a}_2 \cdot \boldsymbol{x}| < 4$, $|\boldsymbol{a}_3 \cdot \boldsymbol{x}| < 6$, $|\boldsymbol{a}_4 \cdot \boldsymbol{x}| < 8$.

**Step 0:**

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{x} = 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{x} = 0$, $\boldsymbol{a}_3 \cdot \boldsymbol{x} = 0$, $\boldsymbol{a}_4 \cdot \boldsymbol{x} = 0$.
- a solution $\boldsymbol{y}_0 = \boldsymbol{0}$.

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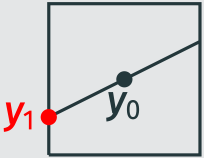
**Step 1:**

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{x} = 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{x} = 0$, $\boldsymbol{a}_3 \cdot \boldsymbol{x} = 0$. (3 eqns, 4 variables)

  in $[-1, 1]^4$ :

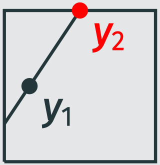



- a solution $\boldsymbol{y}_1 = (\alpha_1, \alpha_2, -1, \alpha_4)$.
- $x_1, x_2, x_4$: floating variables, $x_3 := -1$ is fixed.

**Step 2:**

- Solve $\mathbf{a}_1 \cdot \mathbf{x} = 0$, $\mathbf{a}_2 \cdot \mathbf{x} = 0$. (2 eqns, 3 variables $x_1, x_2, x_4$)

in $[-1, 1]^3$ :




- a solution $\mathbf{y}_2 = (\beta_1, 1, -1, \beta_4)$.
- $x_1, x_4$: floating variables, $x_2 := 1$ is fixed.

**Step 3:**

- Solve $\mathbf{a}_1 \cdot \mathbf{x} = 0$. (1 eqn, 2 variables $x_1, x_4$)
- a solution $\mathbf{y}_3 = (1, 1, -1, \gamma_4)$. ($x_4$ floating, $x_1$ fixed)

## Step 3:

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{x} = 0$. (1 eqn, 2 variables $x_1, x_4$)
- a solution $\boldsymbol{y}_3 = (1, 1, -1, \gamma_4)$. ($x_4$ floating, $x_1$ fixed)

## Step 4:

- choose $x_4 \in \{\pm 1\}$ arbitrarily, say, $\boldsymbol{y}_4 = (1, 1, -1, 1)$.

$$\boldsymbol{y}_4 = (1, 1, -1, 1)$$

$$\boldsymbol{y}_3 = (1, 1, -1, \gamma_4) \qquad \boldsymbol{a}_1 \cdot \boldsymbol{y}_3 = 0 \qquad |\boldsymbol{a}_1 \cdot \boldsymbol{y}_4| < 2$$

$$\boldsymbol{y}_2 = (\beta_1, 1, -1, \beta_4) \qquad \boldsymbol{a}_1 \cdot \boldsymbol{y}_2 = \boldsymbol{a}_2 \cdot \boldsymbol{y}_2 = 0 \qquad |\boldsymbol{a}_2 \cdot \boldsymbol{y}_4| < 4$$

# Multicolor Beck–Spencer

- Beck–Fiala used the floating variable method to study combinatorial discrepancies.
- Benjamin Doerr and Anand Srivastav (2003) extended results of Beck–Fiala to multicolor setting by introducing a multicolor version of the floating variable method.
- We borrow the ideas of Doerr–Srivastav to get a multicolor version of the Beck–Spencer Lemma.
- For simplicity, consider the case $q = 3$.

- Let $H = H(n,3)$ on the vertex set $V = \{1,2,3\}^n$.
- Let $c_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $c_2 = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$, $c_3 = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$.
- Let $Q := \{c_1, c_2, c_3\}$. For $i \in \{1,2,3\}$, assign $c_i$.
- For $v = (v_1, \ldots, v_n) \in V$, assign a $Q$-vector $(c_{v_1}, \ldots, c_{v_n}) \in Q^n$.
- For $v, w \in V$, assign $\boldsymbol{a}, \boldsymbol{x} \in Q^n$, resp. Then $\boldsymbol{a} \cdot \boldsymbol{x} = \frac{2}{3}n - d(v,w)$.

**Lemma (multicolor Beck–Spencer restated)**

For $1 \leq i \leq n$, let $\boldsymbol{a}_i \in Q^n$ be given.
Then there is $\boldsymbol{x} \in Q^n$ such that $|\boldsymbol{a}_i \cdot \boldsymbol{x}| < i$ for all $1 \leq i \leq n$.

We have

- $b(H(n,2)) = \lceil \frac{n}{2} \rceil + 1$,
- $\lfloor \frac{2}{3}n \rfloor + 1 \leq b(H(n,3)) \leq \lfloor \frac{2}{3}n \rfloor + 2$.

**Problem**

Is $b(H(n,3)) = \lfloor \frac{2}{3}n \rfloor + 2$ for $n \geq 3$?

See arXiv:2406.19945 for more details and other problems.