THE MAXIMUM SIZE OF 3-WISE t-INTERSECTING FAMILIES

NORIHIDE TOKUSHIGE

Dedicated to Professor Hikoe Enomoto on the occasion of his sixtieth birthday

ABSTRACT. Let $t \geq 26$ and let $\mathscr{F}$ be a k-uniform hypergraph on n vertices. Suppose that $|F_1 \cap F_2 \cap F_3| \geq t$ holds for all $F_1, F_2, F_3 \in \mathscr{F}$. We prove that the size of $\mathscr{F}$ is at most $\binom{n-t}{k-t}$ if $p = \frac{k}{n}$ satisfies

$$p \leq \frac{2}{\sqrt{4t+9}-1}$$

and n is sufficiently large. The above inequality for p is best possible.

1. INTRODUCTION

A family $\mathscr{F} \subset \binom{[n]}{k}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \geq t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. Let us define r-wise t-intersecting families $\mathscr{F}_i(n,k,r,t)$ as follows:

$$\mathscr{F}_i(n,k,r,t) = \{F \in \binom{[n]}{k} : |F \cap [t+ri]| \geq t + (r-1)i\}.$$

Let $m(n,k,r,t)$ be the maximal size of k-uniform r-wise t-intersecting families on n vertices.

Conjecture 1. $m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|$.

It is known that the conjecture is true for the case $r = 2$, see [1, 2, 4, 6].

Fix $r,t \in \mathbb{N}$ and $p \in \mathbb{Q}$ with $0 < p < 1$. Suppose that $p = \frac{k}{n}$ and let us consider the situation $n \to \infty$ (and hence $k = pn \to \infty$). Writing $\mathscr{F}_i(n,k,r,t)$ as $\mathscr{F}_i$ we have

$$|\mathscr{F}_0| = \binom{n-t}{k-t}, \tag{1}$$

$$|\mathscr{F}_1| = (t+r)\binom{n-(t+r)}{k-(t+r-1)} + \binom{n-(t+r)}{k-(t+r)}, \tag{2}$$

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and

$$\begin{aligned}\lim_{n\to\infty}|\mathscr{F}_0|/\binom{n}{k} &= p^t,\\ \lim_{n\to\infty}|\mathscr{F}_1|/\binom{n}{k} &= (t+r)p^{t+r-1}(1-p)+p^{t+r}\\ &= (t+r)p^{t+r-1}-(t+r-1)p^{t+r}.\end{aligned}$$

Thus $|\mathscr{F}_0|\geq|\mathscr{F}_1|$ (for n large and p fixed) holds iff $p^t\geq(t+r)p^{t+r-1}-(t+r-1)p^{t+r}$, that is,

$$(t+r)p^{r-1}-(t+r-1)p^r-1\leq 0. \tag{3}$$

If $r=2$ then (3) gives $p\leq\frac{1}{t+1}$. In fact $|\mathscr{F}_0(n,k,2,t)|\geq|\mathscr{F}_1(n,k,2,t)|$ holds iff $\frac{k-t+1}{n}\leq\frac{1}{t+1}$. If $r=3$ then (3) gives $p\leq p_t$ where

$$p_t=\frac{2}{\sqrt{4t+9}-1}.$$

The following conjecture is a weaker version (and a special case) of Conjecture 1.

Conjecture 2. *Let $t\in\mathbb{N}$ and $p\in\mathbb{Q}$ be given. Suppose that $t\geq 2$ and $0<p\leq p_t$. Then there exists $n_0(p,t)$ such that $m(n,k,3,t)=\binom{n-t}{k-t}$ holds for $p=\frac{k}{n}$ and $n>n_0(p,t)$.*

If the conjecture is true then the condition on p is sharp. In this paper, we prove the following.

Theorem 1. *Conjecture 2 is true for $t\geq 26$. Moreover, the maximum size $\binom{n-t}{k-t}$ is attained only by $\mathscr{F}_0(n,k,3,t)$ (up to isomorphism).*

Comparing (1) and (2) directly, we have $|\mathscr{F}_0(n,k,3,t)|\geq|\mathscr{F}_1(n,k,3,t)|$ iff

$$n\geq\frac{1}{2}\left(\sqrt{(4t+9)k^2-2(4t^2+11t+3)k+4t^3+13t^2+6t+1}-k+3(t+1)\right), \tag{4}$$

namely, k/n is at most k/R, where R is the RHS of (4). Some computation shows that $k/R>p_t$ for $t\geq 2$ and $k>k_0(t)$, but $k/R\to p_t$ as $k\to\infty$. Thus for $\frac{k}{n}=p_t$ we have $|\mathscr{F}_0(n,k,3,t)|>|\mathscr{F}_1(n,k,3,t)|$ while $|\mathscr{F}_0(n,k,3,t)|/|\mathscr{F}_1(n,k,3,t)|\to 1$ as $k\to\infty$ (and hence $n\to\infty$). Therefore the following result is slightly better than Theorem 1.

Theorem 2. *Let $n,k,t\in\mathbb{N}$ be such that $t\geq 26$, $n>n_0(t)$ and (4). Then we have $m(n,k,3,t)=\binom{n-t}{k-t}$ and equality is attained only by $\mathscr{F}_0(n,k,3,t)$ or $\mathscr{F}_1(n,k,3,t)$ (up to isomorphism).*

Note that R can be an integer, and $\mathscr{F}_1$ is one of the extremal configurations only if $n=R\in\mathbb{N}$.

2. TOOLS

For integers $1 \le i < j \le n$ and a family $\mathscr{F} \subset \binom{[n]}{k}$, define the (i,j)-shift S_{ij} as follows.

$$S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\},$$

where

$$S_{ij}(F) = \begin{cases} (F-\{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ (F-\{j\}) \cup \{i\} \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathscr{F} \subset \binom{[n]}{k}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $1 \le i < j \le n$. For a given family $\mathscr{F}$, one can always obtain a shifted family $\mathscr{F}'$ from $\mathscr{F}$ by applying shifting to $\mathscr{F}$ repeatedly. Then we have $|\mathscr{F}'| = |\mathscr{F}|$ because shifting preserves the size of the family. It is easy to check that if $\mathscr{F}$ is r-wise t-intersecting then $S_{ij}(\mathscr{F})$ is also r-wise t-intersecting. Therefore if $\mathscr{F}$ is an r-wise t-intersecting family then we can find a shifted family $\mathscr{F}'$ which is also r-wise t-intersecting with $|\mathscr{F}'| = |\mathscr{F}|$.

We use the random walk method originated from [3, 4] by Frankl. Let us introduce a partial order in $\binom{[n]}{k}$ by using shifting. For $F, G \in \binom{[n]}{k}$, define $F \succ G$ if G is obtained by repeating a shifting to F. The following fact follows immediately from definition.

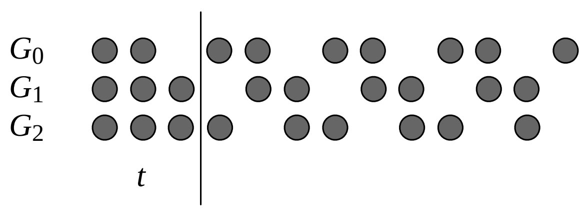
Fact 1. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted family. If $F \in \mathscr{F}$ and $F \succ G$, then $G \in \mathscr{F}$.*

For $F \in \binom{[n]}{k}$ we define the corresponding walk on $\mathbb{Z}^2$, denoted by walk(F), in the following way. The walk is from $(0,0)$ to $(n-k,k)$ with n steps, and if $i \in F$ (resp. $i \notin F$) then the i-th step is one unit up (resp. one unit to the right).

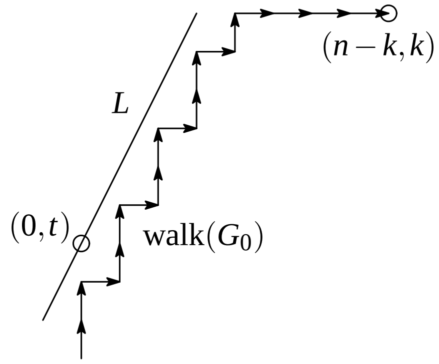
Fact 2 ([3]). *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. Then for all $F \in \mathscr{F}$, walk(F) must touch the line $L : y = (r-1)x + t$.*

Proof. We only prove the case $r = 3$ (but one can prove the general case in exactly the same way). Let $i_0 = \lfloor \frac{k-t}{2} \rfloor$, $i_1 = \lfloor \frac{k-t-1}{2} \rfloor$ and set

$$\begin{aligned} G_0 &= [t-1] \cup \{t+3i+1 : 0 \le i \le i_0\} \cup \{t+3i+2 : 0 \le i \le i_1\}, \\ G_1 &= [t-1] \cup \{t+3i : 0 \le i \le i_0\} \cup \{t+3i+2 : 0 \le i \le i_1\}, \\ G_2 &= [t-1] \cup \{t+3i : 0 \le i \le i_0\} \cup \{t+3i+1 : 0 \le i \le i_1\}. \end{aligned}$$



Assume that $G_0 \in \mathscr{F}$. Since $G_0 \succ G_1 \succ G_2$ we also have $G_1, G_2 \in \mathscr{F}$ by Fact 1. But this is impossible because $G_0 \cap G_1 \cap G_2 = [t-1]$, which contradicts the 3-wise t-intersecting property of $\mathscr{F}$. Thus we must have $G_0 \notin \mathscr{F}$. Note that G_0 is the "minimal" set (in the shifting order poset) whose corresponding walk does not touch the line $L : y = 2x + t$.



Thus if $F \in \mathscr{F}$ and walk(F) does not touch the line, then we have $F \succ G_0$, and by Fact 1 we have $G_0 \in \mathscr{F}$, which is a contradiction. □

The next result (Corollary 8 and Theorem 4 in [5]) enables us to upper bound the number of walks which touch a given line.

Proposition 1. *Let $p \in \mathbb{Q}$, $r, t, u, v \in \mathbb{N}$ be fixed constants and let $n, k \in \mathbb{N}$ with $p = \frac{k}{n}$, $p < \frac{r-1}{r+1}$ and $r \geq 2$. Let $\alpha \in (p, 1)$ be the unique root of the equation $(1-p)x^r - x + p = 0$ and let $g(n)$ be the number of walks from (u, v) to $(n-k, k)$ which touch the line $y = (r-1)(x-u) + v + s$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{g(n)}{\binom{n-u-v}{k-v}} \leq (1+\varepsilon)\alpha^s$$

holds for all $n > n_0$. Moreover if $u = 0$ then we can choose $\varepsilon = 0$.

3. Proof of Theorem 1

We first prove the theorem for $t \geq 75$ in subsection 3.1, where all the basic ideas are included. Then in subsection 3.2 we improve the lower bound for t using more detailed casewise analysis.

3.1. **Proof for $t \geq 75$.** Let $p \in \mathbb{Q}$ with $0 < p \leq p_t (= \frac{2}{\sqrt{4t+9}-1})$ be given. Set $q = 1 - p$ and $\alpha = \alpha_p = \frac{1}{2}(\sqrt{\frac{1+3p}{1-p}} - 1)$. Note that $\alpha \in (p, 1)$ is the root of the equation $(1-p)x^3 - x + p = 0$.

Let $p = \frac{k}{n}$ and let $\mathscr{H} \subset \binom{[n]}{k}$ be a shifted 3-wise t-intersecting family. Then by Fact 2 walk(H) hits the line $L : y = 2x + t$ for all $H \in \mathscr{H}$. Thus by Proposition 1 (setting $u = v = 0$, $s = t$) we have $|\mathscr{H}| \le \alpha^t \binom{n}{k}$. Our goal is to prove that $|\mathscr{H}| < \binom{n-t}{k-t}$ unless $\mathscr{H} \cong \mathscr{F}_0(n,k,3,t)$.

For $0 \le i \le \lfloor \frac{k-t}{2} \rfloor$ let us define

$$\mathscr{G}_i = \{G \in \binom{[n]}{k} : |G \cap [t+3\ell]| \ge t + 2\ell \text{ first holds at } \ell = i\}.$$

In other words, $G \in \mathscr{G}_i$ iff walk(G) reaches the line L at $(i, t+2i)$ for the first time. Set $\mathscr{H}_i = \mathscr{H} \cap \mathscr{G}_i$. For an infinite set $A = \{a_1, a_2, \dots\} \subset \mathbb{N}$ with $a_1 < a_2 < \cdots$, let us define $\text{First}_k(A) = \{a_1, a_2, \dots, a_k\}$. Set

$$\begin{aligned} T(i) &= \{i, i+3, i+6, \dots\} = \{i + 3j : j \ge 0\}, \\ A_i^* &= [t] \cup \{t+i+1\} \cup T(t+i+3) \cup T(t+i+4), \\ B_i^* &= [t-1] \cup \{t+1, t+2, t+3\} \cup T(t+i+4) \cup T(t+i+6), \end{aligned}$$

and $A_i = \text{First}_k(A_i^*)$, $B_i = \text{First}_k(B_i^*)$. We will use only small i so that $A_i, B_i \in \binom{[n]}{k}$, and then we have $A_i \in \mathscr{G}_0$ and $B_i \in \mathscr{G}_1$.

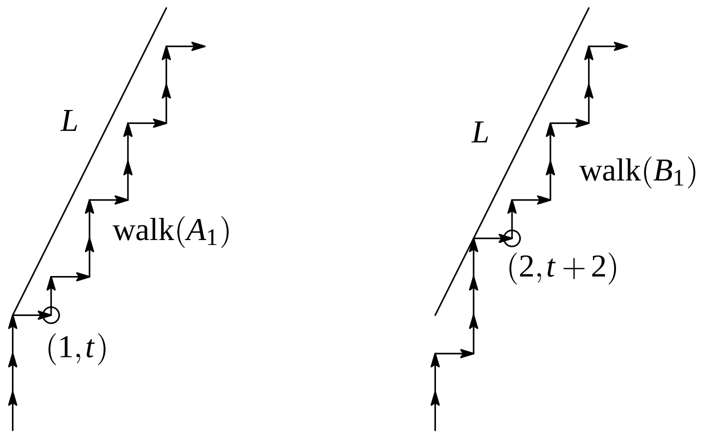
We consider three cases according to the structure of $\mathscr{H}$. If $\mathscr{H}$ is (somewhat) similar to $\mathscr{F}_0(n,k,3,t)$ then we compare $\mathscr{H}$ with $\mathscr{F}_0(n,k,3,t)$ and this is Case 2. In Case 3 we compare $\mathscr{H}$ with $\mathscr{F}_1(n,k,3,t)$. If $\mathscr{H}$ is neither similar to $\mathscr{F}_0$ nor $\mathscr{F}_1$ then it is less likely that $\mathscr{H}$ has large size, but in this case we do not have an appropriate comparison object, which makes it difficult to bound the size of $\mathscr{H}$. We deal with this situation in Case 1, and we will refine the estimation for this case in the next subsection again.

Case 1. $A_1 \notin \mathscr{H}$ and $B_1 \notin \mathscr{H}$.
Suppose that $H \in \mathscr{H}_0$. Then after passing the point $(0,t)$, walk(H) goes to $(0, t+1)$ or $(1,t)$. So we can divide $\mathscr{H}_0 = \mathscr{H}_0^{(0,t+1)} \cup \mathscr{H}_0^{(1,t)}$ according to the next point to $(0,t)$ in the walk. For $\mathscr{H}_0^{(0,t+1)}$ we use a trivial bound

$$|\mathscr{H}_0^{(0,t+1)}| \le \binom{n-(t+1)}{k-(t+1)} \approx p^{t+1}\binom{n}{k}, \tag{5}$$

where we denote $a \approx b$ iff $\lim_{n\to\infty} a/b = 1$. If $H \in \mathscr{H}_0^{(1,t)}$ then walk(H) must touch the line L after passing $(1,t)$. Otherwise we get $H \succ A_1$, which means $H \notin \mathscr{H}$, a contradiction. Here we used the fact that A_1 is the minimal set (in the shifting order poset) whose walk does not touch the line L after passing $(1,t)$. Thus by Proposition 1 (setting $u = 1$, $v = t$, $s = 2$) we have

$$|\mathscr{H}_0^{(1,t)}| \le (1+\varepsilon)\alpha^2\binom{n-(t+1)}{k-t} \approx \alpha^2 p^t q \binom{n}{k}. \tag{6}$$



Next suppose that $H \in \mathscr{H}_1$. Then after passing $(1,t+2)$, walk(H) goes to $(1,t+3)$ or $(2,t+2)$. So we can divide $\mathscr{H}_1 = \mathscr{H}_1^{(1,t+3)} \cup \mathscr{H}_1^{(2,t+2)}$. Noting that there are t ways of walking from $(0,0)$ to $(1,t+3)$ which avoid passing $(0,t)$, we have

$$|\mathscr{H}_1^{(1,t+3)}| \leq t\binom{n-(t+4)}{k-(t+3)} \approx tp^{t+3}q\binom{n}{k}. \tag{7}$$

If $H \in \mathscr{H}_1^{(2,t+2)}$, then walk$(H)$ must touch L after passing $(2,t+2)$. Otherwise we get $H \succ B_1$, which means $H \notin \mathscr{H}$, a contradiction. Thus by Proposition 1 (setting $u=2$, $v=t+2$, $s=2$) we have

$$|\mathscr{H}_1^{(2,t+2)}| \leq (1+\varepsilon)t\alpha^2\binom{n-(t+4)}{k-(t+2)} \approx (1+\varepsilon)t\alpha^2 p^{t+2}q^2\binom{n}{k}. \tag{8}$$

Finally we count the number of H in $\bigcup_{i\geq 2}\mathscr{H}_i \subset \bigcup_{i\geq 2}\mathscr{G}_i$. Then we have

$$\begin{aligned} |\bigcup_{i\geq 2}\mathscr{H}_i| &\leq |\bigcup_{i\geq 0}\mathscr{G}_i| - |\mathscr{G}_0| - |\mathscr{G}_1| \\ &\leq \alpha^t\binom{n}{k} - \binom{n-t}{k-t} - t\binom{n-(t+3)}{k-(t+2)} \\ &\approx (\alpha^t - p^t - tp^{t+2}q)\binom{n}{k}. \end{aligned} \tag{9}$$

Therefore by (5), (6), (7), (8) and (9) we have

$$\frac{|\mathscr{H}|}{\binom{n}{k}} \leq (1+o(1))(p^{t+1}+\alpha^2p^tq+tp^{t+3}q+t\alpha^2p^{t+2}q^2+\alpha^t-p^t-tp^{t+2}q) \tag{10}$$

as $n\to\infty$. On the other hand we have $\binom{n-t}{k-t} \approx p^t\binom{n}{k}$. Consequently it suffices to show that

$$p^{t+1}+\alpha^2p^tq+tp^{t+3}q+t\alpha^2p^{t+2}q^2+\alpha^t-p^t-tp^{t+2}q < p^t, \tag{11}$$

or equivalently,

$$(\alpha/p)^t - t(1-\alpha^2)p^2q^2 + \alpha^2 q + p - 2 < 0. \tag{12}$$

Since the LHS is an increasing function of t, it suffices to show the inequality for $p = p_t$ and this is true for $t \geq 75$. In fact we can find $\gamma > 0$ such that the LHS of (12) is less than $-\gamma$, or equivalently, the LHS of (11) is less than $(1-\gamma)p^t$. See Appendix for more details. Thus by (10) we have

$$|\mathscr{H}| \leq (1+o(1))(1-\gamma)p^t \binom{n}{k} < \binom{n-t}{k-t}$$

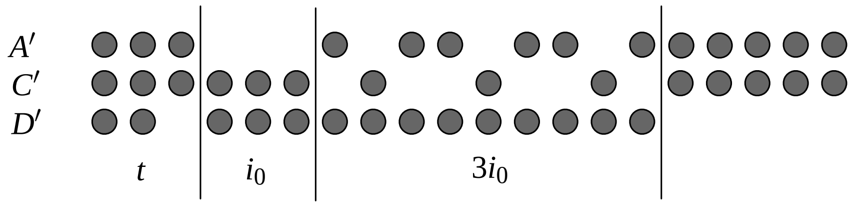
for $n > n_0(p,t)$ and $t \geq 75$.

Case 2. $A_1 \in \mathscr{H}$.
If $[t] \subset H$ holds for all $H \in \mathscr{H}$ then it follows that $|\mathscr{H}| \leq \binom{n-t}{k-t}$ and equality holds iff $\mathscr{H} \cong \mathscr{F}_0(n,k,3,t)$. Thus we may assume that $[t] \not\subset H$ holds for some $H \in \mathscr{H}$ and in particular since $\mathscr{H}$ is shifted we may assume that $D' = [k+1] - \{t\} \in \mathscr{H}$.

Let $i_0 = \lceil \frac{k+1-t}{4} \rceil$ and set

$$\begin{aligned} \tilde{A} &= [t] \cup (\bigcup_{j=0}^{i_0-1} \{t+i_0+3j+1, t+i_0+3j+3\}) \cup \{t+4i_0+j : j \geq 1\}, \\ \tilde{C} &= [t+i_0] \cup \{t+i_0+3j+2 : 0 \leq j \leq i_0-1\} \cup \{t+4i_0+j : j \geq 1\}, \end{aligned}$$

and let $A' = \mathrm{First}_k(\tilde{A})$, $C' = \mathrm{First}_k(\tilde{C})$.

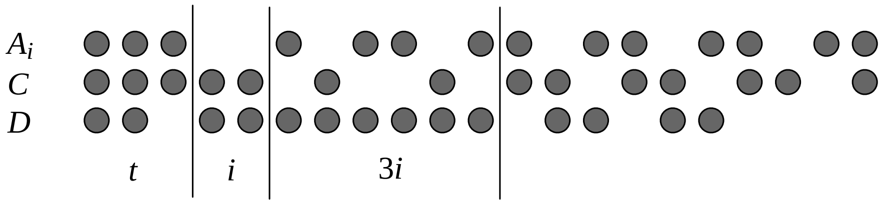


Suppose that $A' \in \mathscr{H}$. Then $A' \succ C'$ implies that $C' \in \mathscr{H}$. Since $4i_0 + t \geq k+1$ we have $A' \cap C' \cap D' = [t-1]$ but this is impossible because $\mathscr{H}$ is 3-wise t-intersecting. Thus we have $A' \notin \mathscr{H}$, and since $A_i \succ A'$ for $i \geq i_0$ we also have $A_i \notin \mathscr{H}$ if $i \geq i_0$.

Now let $1 \leq i < i_0$ be such that $A_i \in \mathscr{H}$ but $A_{i+1} \notin \mathscr{H}$. Let

$$\begin{aligned} C^* &= [t+i] \cup \{t+i+3j+2 : 0 \leq j < i\} \\ &\quad \cup (\bigcup_{j \geq 0} \{t+4i+3j+1, t+4i+3j+2\}), \\ D^* &= [t-1] \cup [t+1, t+4i] \cup (\bigcup_{j \geq 0} \{t+4i+3j+2, t+4i+3j+3\}), \end{aligned}$$

and let $C = \mathrm{First}_k(C^*)$, $D = \mathrm{First}_k(D^*)$.

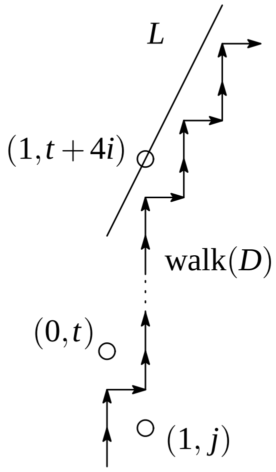


Then we have $C \in \mathscr{H}$ because $A_i \in \mathscr{H}$ and $A_i \succ C$. Since $\mathscr{H}$ is 3-wise t-intersecting and $A_i \cap C \cap D = [t-1]$ we can conclude that $D \notin \mathscr{H}$.

Let $H \in \mathscr{H}$. First suppose that walk(H) does not pass $(0,t)$, i.e., $H \cap [t] \neq [t]$. Then walk(H) must go through (at least) one of the points in

$$P = \{(1,0),(1,1),\dots,(1,t-1)\}.$$

Let $(1,j)$ $(0 \le j \le t-1)$ be the first point in P that walk(H) hits. In other words, we have $H \cap [j+1] = [j]$. From the point $(1,j)$, walk(H) must touch the line $L : y = 2(x-1)+t+4i$, otherwise we get $H \succ D$ and $D \in \mathscr{H}$, which is a contradiction.



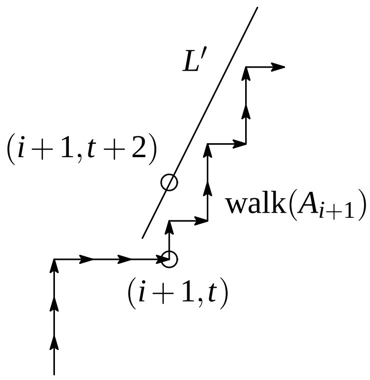
We estimate the number of walks from $(1,j)$ to $(n-k,k)$ which touch the line L. By Proposition 1 (setting $u=1$, $v=j$, $s=t+4i-j$) the number is at most

$$(1+\varepsilon)\alpha^{t+4i-j}\binom{n-(j+1)}{k-j}.$$

Therefore the number of $H \in \mathscr{H}$ such that $H \cap [t] \neq [t]$ is at most

$$(1+\varepsilon)\sum_{j=0}^{t-1}\alpha^{t+4i-j}\binom{n-(j+1)}{k-j}. \tag{13}$$

Next suppose that walk(H) passes $(0,t)$, i.e., $H \cap [t] = [t]$. The number of corresponding walks is at most $\binom{n-t}{k-t}$, but we need to refine this estimation. Suppose that walk(H) passes $(i+1,t)$. Then from this point walk(H) must touch the line $L' : y = 2x + t - 2i$, otherwise we get $H \succ A_{i+1}$ and $A_{i+1} \in \mathscr{H}$, which is a contradiction.



The trivial upper bound for the number of walks from $(i+1,t)$ to $(n-k,k)$ is $\binom{n-(t+i+1)}{k-t}$, but those walks in $\mathscr{H}$ touch the line L' and so by Proposition 1 we will get an improved upper bound for the number of walks of this type. To apply the proposition, it is convenient to neglect the first $i+t+1$ steps of the walks, in other words, we shift the origin to $(i+1,t)$, and replace n and k by $n' = n-(t+i+1)$ and $k' = k-t$. Then L' becomes $y = 2x+2$ in the new coordinates, and by setting $u = v = 0$ and $s = 2$, Proposition 1 gives an improved upper bound $\alpha_{p'}^2\binom{n'}{k'}$ where $p' = \frac{k'}{n'} \approx \frac{k}{n-i}$ and $\alpha_{p'} = \frac{1}{2}(\sqrt{\frac{1+3p'}{1-p'}}-1)$. Therefore the number of $H \in \mathscr{H}$ such that $H \cap [t] = [t]$ is at most

$$\binom{n-t}{k-t} - (1-\alpha_{p'}^2)\binom{n'}{k'}. \tag{14}$$

We shall show $|\mathscr{H}| < \binom{n-t}{k-t}$. By (13) and (14) it suffices to prove that

$$(1+\varepsilon)\sum_{j=0}^{t-1}\alpha^{t+4i-j}\binom{n-(j+1)}{k-j} - (1-\alpha_{p'}^2)\binom{n'}{k'} < 0,$$

or equivalently,

$$\sum_{j=0}^{t-1}\alpha^{t-j}\binom{n-(j+1)}{k-j} < \frac{1-\alpha_{p'}^2}{(1+\varepsilon)\alpha^{4i}}\binom{n'}{k'} := f(i). \tag{15}$$

Claim 1. *$f(i)$ is an increasing function of i.*

Proof. To show $f(i-1) < f(i)$, let $p'' = \frac{k-t}{n-t-(i-1)-1} = \frac{k'}{n'+1}$. Then we need to show

$$\frac{1-\alpha_{p''}^2}{(1+\varepsilon)\alpha^{4(i-1)}}\binom{n'+1}{k'} < \frac{1-\alpha_{p'}^2}{(1+\varepsilon)\alpha^{4i}}\binom{n'}{k'},$$

which is equivalent to

$$\frac{1-\alpha_{p''}^2}{1-\alpha_{p'}^2} < \frac{1}{\alpha^4}\binom{n'}{k'}\Big/\binom{n'+1}{k'} = \frac{1}{\alpha^4}\cdot\frac{n'+1-k}{n'+1}.$$

We have $\frac{n'+1-k}{n'+1} = \frac{n-k-i}{n-t-i} \geq \frac{n-k-i_0}{n-t-i_0} \approx (1-\frac{5}{4}p)/(1-\frac{p}{4})$ and some computation shows $\alpha^4 < (1-\frac{5}{4}p)/(1-\frac{p}{4})$ for $p < 0.55$. Thus we can choose $\delta > 0$ so small that

$$1+\delta < \frac{1}{\alpha^4}\cdot\frac{n'+1-k}{n'+1}$$

holds for $n > n_0(\delta)$ and $p < 0.55$. On the other hand, since $\frac{1}{p''} = \frac{1}{p'} + \frac{1}{k'}$ we have $p'' \approx p'$ and hence

$$\frac{1-\alpha_{p''}^2}{1-\alpha_{p'}^2} < 1+\delta$$

for $n > n_1(\delta)$. □

Thus it suffices to show (15) for $i = 1$. Noting that $p' \approx p$, $\binom{n-(j+1)}{k-j} \approx p^j q\binom{n}{k}$ and $\binom{n-(t+2)}{k-t} \approx p^t q^2\binom{n}{k}$, we find that the target inequality is

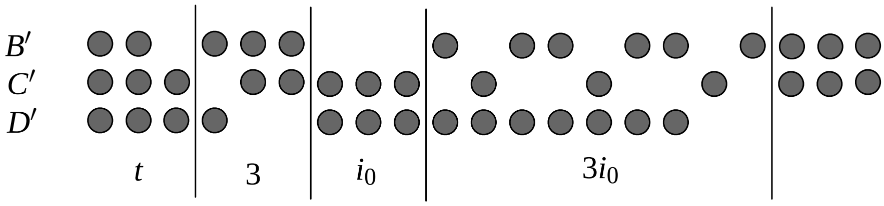
$$(\alpha/p)^t - 1 < \frac{1-\alpha^2}{\alpha^4}q\big(1-(p/\alpha)\big).$$

The LHS is an increasing function of t, and for $p = p_t$ one can verify that the inequality is true for $t \geq 8$.

Case 3. $B_1 \in \mathscr{H}$.
Let $D' = [k+2] - \{t+2, t+3\}$. If $D' \notin \mathscr{H}$ then the shiftedness of $\mathscr{H}$ implies that $\mathscr{H} \subset \mathscr{F}_1(n,k,3,t)$ and we are done. (Recall that we have $|\mathscr{F}_1(n,k,3,t)| < |\mathscr{F}_0(n,k,3,t)| = \binom{n-t}{k-t}$ for $0 < p \leq p_t$.) Thus we may assume that $D' \in \mathscr{H}$. Let $i_0 = \lceil \frac{k-t-1}{4} \rceil$ and set

$$\begin{aligned}
\tilde{B} &= ([t+3]-\{t\}) \cup \Big(\bigcup_{j=0}^{i_0-1}\{(t+3+i_0)+3j+1, (t+3+i_0)+3j+3\}\Big) \\
&\quad \cup \{t+3+4i_0+j : j \geq 1\}, \\
\tilde{C} &= ([t+3+i_0]-\{t+1\}) \cup \{(t+3+i_0)+3j+2 : 0 \leq j \leq i_0-1\} \\
&\quad \cup \{t+3+4i_0+j : j \geq 1\},
\end{aligned}$$

and let $B' = \mathrm{First}_k(\tilde{B})$, $C' = \mathrm{First}_k(\tilde{C})$.

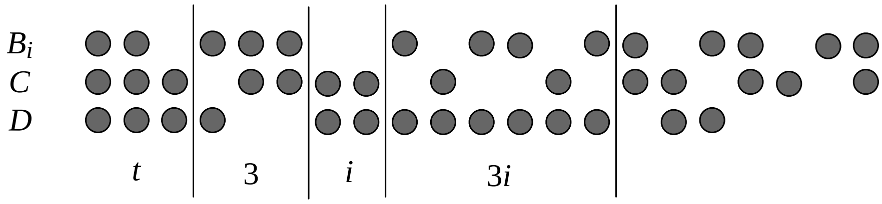


Suppose that $B' \in \mathscr{H}$. Then $B' \succ C'$ implies that $C' \in \mathscr{H}$. Since $t+3+4i_0 \geq k+2$ we have $B' \cap C' \cap D' = [t-1]$, which contradicts the 3-wise t-intersecting property of $\mathscr{H}$. Thus we have $B' \notin \mathscr{H}$ and also $B_i \notin \mathscr{H}$ for $i \geq i_0$.

Now let $1 \leq i < i_0$ be such that $B_i \in \mathscr{H}$ but $B_{i+1} \notin \mathscr{H}$. Let

$$\begin{aligned} C^* &= ([t+3+i]-\{t+1\}) \cup \{(t+3+i)+3j+2 : 0 \leq j < i\} \\ &\quad \cup (\bigcup_{j \geq 0} \{(t+3+4i)+3j+1, (t+3+4i)+3j+2\}), \\ D^* &= ([t+3+4i]-\{t+2, t+3\}) \\ &\quad \cup (\bigcup_{j \geq 0} \{(t+3+4i)+3j+2, (t+3+4i)+3j+3\}). \end{aligned}$$

and let $C = \mathrm{First}_k(C^*)$, $D = \mathrm{First}_k(D^*)$.

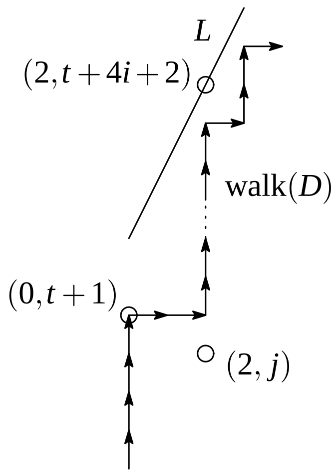


Then we have $C \in \mathscr{H}$ because $B_i \in \mathscr{H}$ and $B_i \succ C$. Since $\mathscr{H}$ is 3-wise t-intersecting and $B_i \cap C \cap D = [t-1]$ we can conclude that $D \notin \mathscr{H}$.

Let $H \in \mathscr{H}$. First suppose that walk(H) passes (at least) one of the points in $P = \{(2,0), (2,1), \ldots, (2,t+1)\}$, i.e., $|H \cap [t+3]| \leq t+1$. Let $(2, j)$ $(0 \leq j \leq t+1)$ be the first point in P that walk(H) hits. From this point, walk(H) must touch the line $L : y = 2(x-2)+t+4i+2$, otherwise we get $H \succ D$ and $D \in \mathscr{H}$, a contradiction. Thus the number of corresponding walks is at most

$$(j+1)(1+\varepsilon)\alpha^{t+4i+2-j}\binom{n-(j+2)}{k-j},$$

where $j+1$ is the number of walks from $(0,0)$ to $(2, j)$ which do not touch $\{(2,\ell) : 0 \leq \ell < j\}$.



Hence the number of $H \in \mathscr{H}$ such that $|H \cap [t+3]| \leq t+1$ is at most

$$(1+\varepsilon)\sum_{j=0}^{t+1}(j+1)\alpha^{t+4i+2-j}\binom{n-(j+2)}{k-j}. \tag{16}$$

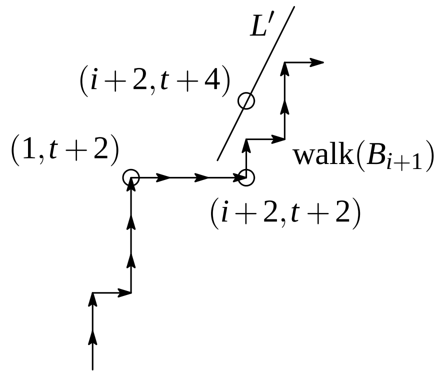
Next suppose that $|H \cap [t+3]| \geq t+2$. Then walk(H) passes $(0,t+3)$ or $(1,t+2)$. The number of walks which pass $(0,t+3)$ is at most

$$\binom{n-(t+3)}{k-(t+3)}. \tag{17}$$

The number of walks which pass $(1,t+2)$ is clearly at most $(t+3)\binom{n-(t+3)}{k-(t+2)}$ and we will improve this estimation. Suppose that walk(H) passes $(1,t-1)$, $(1,t+2)$ and $(i+2,t+2)$. Then from $(i+2,t+2)$, this walk must touch the line $L' : y = 2(x-(i+2))+t+4$, otherwise we get $H \succ B_{i+1}$ and $B_{i+1} \in \mathscr{H}$, a contradiction. Thus the number of walks in $\mathscr{H}$ which pass $(1,t+2)$ is at most

$$(t+3)\binom{n-(t+3)}{k-(t+2)} - t\left(1-\alpha_{p'}^2\right)\binom{n'}{k'}, \tag{18}$$

where $n' = n-t-i-4$, $k' = k-t-2$ and $p' = \frac{k'}{n'} \approx \frac{k}{n-i}$.



We shall show that the sum of (16), (17) and (18) is less than $|\mathscr{F}_1(n,k,3,t)| = (t+3)\binom{n-(t+3)}{k-(t+2)} + \binom{n-(t+3)}{k-(t+3)}$, which means $|\mathscr{H}| < |\mathscr{F}_1|$. Our target inequality is

$$(1+\varepsilon)\sum_{j=0}^{t+1}(j+1)\alpha^{t+2-j}\binom{n-(j+2)}{k-j} < \frac{t}{\alpha^{4i}}(1-\alpha_{p'}^2)\binom{n'}{k'}.$$

The RHS is an increasing function of i. (One can show this fact similarly to the proof of Claim 1.) Thus we show the inequality for $i=1$. Consequently it suffices to show

$$\frac{\alpha^4}{t(1-\alpha^2)q}\sum_{j=0}^{t+1}(j+1)(\alpha/p)^{t+2-j} < 1.$$

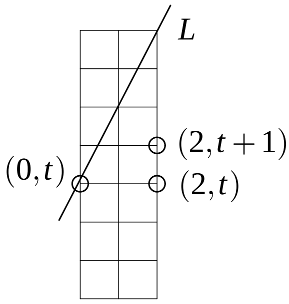
Noting that α/p is an increasing function of p we find that the LHS is an increasing function of p. Then with some routine computation one can check that for $p=p_t$ the inequality is true if $t \geq 7$.

3.2. **Further improvement.** In the previous subsection, we proved the theorem for $t \geq 75$ ($t \geq 75$ in Case 1, $t \geq 8$ in Case 2 and $t \geq 7$ in Case 3). Here we will refine the proof for Case 1, and will prove the theorem for $t \geq 26$.

Assume that $A_1 \notin \mathscr{H}$ and $B_1 \notin \mathscr{H}$. Let

$$\begin{aligned}\tilde{\mathscr{H}}_0^{(0,t+1)} &= \{H-[t+1] : [t+1] \subset H \in \mathscr{H}\},\\ \tilde{\mathscr{H}}_1^{(1,t+3)} &= \{H\cap[t+5,n] : H \in \mathscr{H}_1^{(1,t+3)}\}.\end{aligned}$$

Case 1a. $\tilde{\mathscr{H}}_0^{(0,t+1)}$ is not 2-wise 1-intersecting.
In this case we have $G,G' \in \mathscr{H}$ such that $G\cap G' = [t+1]$. Let $H \in \mathscr{H}$. Since $\mathscr{H}$ is 3-wise t-intersecting we have $|H\cap[t+1]| \geq t$. Thus walk(H) hits $(0,t+1)$ or $(1,t)$, and walk(H) never hits a point in $\{(2,0),(2,1),\ldots,(2,t-1)\}$. In particular, if $H \in \bigcup_{i\geq 2}\mathscr{H}_i$ then walk(H) reaches the line $x=2$ for the first time only at $(2,t)$ or $(2,t+1)$. In both cases walk(H) passes $(1,t)$ and there are t ways of walking from $(0,0)$ to $(1,t)$ which avoid $(0,t)$.



Then after passing $(2,t)$ or $(2,t+1)$, walk(H) must touch the line $L: y = 2x+t$. Therefore we have

$$\begin{aligned} |\bigcup_{i\geq 2} \mathscr{H}_i| &\leq (1+\varepsilon)\left(t\alpha^4 \binom{n-(t+2)}{k-t} + t\alpha^3 \binom{n-(t+3)}{k-(t+1)} \right) \\ &\approx t\alpha^3(\alpha+p)p^t q^2 \binom{n}{k}. \end{aligned} \tag{19}$$

By (5), (6), (7), (8) and (19) it suffices to show that

$$p^{t+1} + \alpha^2 p^t q + tp^{t+3} q + t\alpha^2 p^{t+2} q^2 + t\alpha^3(\alpha+p)p^t q^2 < p^t,$$

and this is true for $t \geq 8$ and $0 < p \leq p_t$.

Case 1b. Both $\tilde{\mathscr{H}}_0^{(0,t+1)}$ and $\tilde{\mathscr{H}}_1^{(1,t+3)}$ are 2-wise 1-intersecting.
In this case we use the (simplest) Erdős–Ko–Rado Theorem to bound the sizes of $\mathscr{H}_0^{(0,t+1)}$ and $\mathscr{H}_1^{(1,t+3)}$. Then we have

$$|\mathscr{H}_0^{(0,t+1)}| = |\tilde{\mathscr{H}}_0^{(0,t+1)}| \leq \binom{n-(t+1)-1}{k-(t+1)-1} \approx p^{t+2}\binom{n}{k}, \tag{20}$$

$$|\mathscr{H}_1^{(1,t+3)}| = t|\tilde{\mathscr{H}}_1^{(1,t+3)}| \leq t\binom{n-(t+4)-1}{k-(t+3)-1} \approx tp^{t+4}q\binom{n}{k}. \tag{21}$$

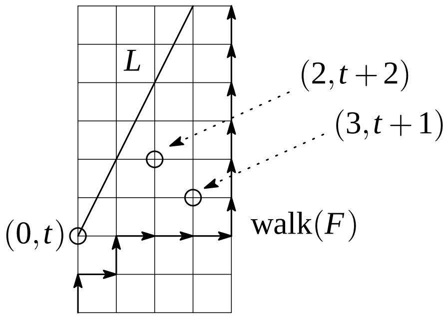
Therefore by (20), (6), (21), (8) and (9) it suffices to show that

$$p^{t+2} + \alpha^2 p^t q + tp^{t+4} q + t\alpha^2 p^{t+2} q^2 + \alpha^t - p^t - tp^{t+2} q < p^t,$$

and this is true for $t \geq 18$ and $0 < p \leq p_t$.

Case 1c. $\tilde{\mathscr{H}}_0^{(0,t+1)}$ is 2-wise 1-intersecting and $\tilde{\mathscr{H}}_1^{(1,t+3)}$ is not 2-wise 1-intersecting. We use (20) to bound $\mathscr{H}_0^{(0,t+1)}$ again. Now we will bound the size of $\bigcup_{i\geq 2} \mathscr{H}_i$. Since $\tilde{\mathscr{H}}_1^{(1,t+3)}$ is not 2-wise 1-intersecting and $\mathscr{H}$ is shifted, we have $G, G' \in \mathscr{H}$ such that $G \cap G' = [t+4] - \{t\}$. If $F = [k+4] - \{t, t+2, t+3, t+4\} \in \mathscr{H}$ then we also have $F' = [k+4] - \{t+1, t+2, t+3, t+4\} \in \mathscr{H}$ by shifting. But this is impossible because $G \cap G' \cap F' = [t-1]$. Thus we must have $F \notin \mathscr{H}$. Let

$H \in \bigcup_{i\geq 2}\mathscr{H}_i$. Then walk$(H)$ never hits a point in $\{(4,0),(4,1),\ldots,(4,t)\}$, otherwise we get $H \succ F \in \mathscr{H}$, a contradiction. In other words, walk(H) passes $(2,t+2)$ or $(3,t+1)$.



There are $u_1 = \binom{t+1}{2} + 2\binom{t}{1}$ ways (resp. $u_2 = \binom{t+2}{3} + 2\binom{t+1}{2} + 3\binom{t}{1}$ ways) of walking from $(0,0)$ to $(2,t+2)$ (resp. from $(0,0)$ to $(3,t+1)$) which do not touch the line $L : y = 2x+t$. Then after passing $(2,t+2)$ or $(3,t+1)$, walk(H) must touch the line L. Therefore we have

$$\begin{aligned} |\bigcup_{i\geq 2}\mathscr{H}_i| &\leq (1+\varepsilon)\left(u_1\alpha^2\binom{n-(t+4)}{k-(t+2)} + u_2\alpha^5\binom{n-(t+4)}{k-(t+1)}\right) \\ &\approx (u_1 p + u_2\alpha^3 q)\alpha^2 p^{t+1} q^2 \binom{n}{k}. \end{aligned} \tag{22}$$

Consequently by (20), (6), (7), (8) and (22) it suffices to show that

$$p^{t+2} + \alpha^2 p^t q + t p^{t+3} q + t\alpha^2 p^{t+2} q^2 + (u_1 p + u_2 \alpha^3 q)\alpha^2 p^{t+1} q^2 < p^t,$$

and this is true for $t \geq 26$ and $0 < p \leq p_t$.

4. Proof of Theorem 2

The proof of Theorem 2 is almost identical to the proof of Theorem 1. The only difference is that we assume (4) instead of assuming $0 < p \leq p_t$ where $p = k/n$.

In Case 1, we can choose $\delta > 0$ sufficiently small so that (12) holds for $0 < p < p_t + \delta$. If $n > n_0(t)$ then we may assume that $k/n < p_t + \delta$. Thus the remaining part goes through without changes. (We only need to change γ a little bit smaller.)

Similarly in Case 2 we can check that $\mathscr{H} \subset \mathscr{F}_0(n,k,3,t)$ or $|\mathscr{H}| < \binom{n-t}{k-t}$. Also in Case 3 we can show that $\mathscr{H} \subset \mathscr{F}_1(n,k,3,t)$ or $|\mathscr{H}| < |\mathscr{F}_1(n,k,3,t)|$. Case 1a, Case 1b and Case 1c are similar to Case 1, and we omit the details.

5. Appendix

Here we give an outline of proof of (12), namely

$$f(p,t) = (\alpha/p)^t - t(1-\alpha^2)p^2q^2 + \alpha^2 q + p - 2 < 0$$

for $0 < p \leq p_t$, $\alpha = \frac{1}{2}(\sqrt{\frac{1+3p}{1-p}} - 1)$ and $t \geq 75$. First we check that $f(p,t)$ is an increasing function of t. It suffices to show $\frac{\partial f}{\partial t}(p,t) > 0$, or equivalently,

$$(\alpha/p)^t > \frac{(1-\alpha^2)p^2q^2}{\log(\alpha/p)}.$$

In fact, one can show (with some computation) that the RHS is at most 1, while the LHS is clearly more than 1 because $\alpha > p$.

Now we assume that $p = p_t$. Setting $t = \frac{1}{x^2}$, the Taylor expansion of $f(p,t)$ at $x = 0$ gives

$$f(p,t) = -3 + e + 2x + \big(6 - \frac{e}{2}\big)x^2 - \big(\frac{15}{4} + e\big)x^3 - \big(17 - \frac{47e}{24}\big)x^4 + Rx^5,$$

where $e = \exp(1)$ and $0 \leq R \leq 26$ for $0 < x < 0.116$. Thus we have $f(p,t) < 0$ if $0 < x < 0.1156$, i.e., $t = \frac{1}{x^2} \geq 74.83$, moreover we have $f(p,t) < -\gamma$ for $\gamma = 0.0004$ and $t \geq 75$.

The inequalities in the other cases can be proved similarly, and we omit the technical details.

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College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213 Japan

E-mail address: `hide@edu.u-ryukyu.ac.jp`