EKR TYPE INEQUALITIES FOR 4-WISE INTERSECTING FAMILIES

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ABSTRACT. Let $1 \le t \le 7$ be an integer and let $\mathscr{F}$ be a k-uniform hypergraph on n vertices. Suppose that $|A \cap B \cap C \cap D| \ge t$ holds for all $A,B,C,D \in \mathscr{F}$. Then we have $|\mathscr{F}| \le \binom{n-t}{k-t}$ if $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ holds for some $\varepsilon > 0$ and all $n > n_0(\varepsilon)$. We apply this result to get EKR type inequalities for "intersecting and union families" and "intersecting Sperner families."

1. INTRODUCTION

A family $\mathscr{F} \subset 2^{[n]}$ is called r-wise t-intersecting if $|F_1 \cap \cdots \cap F_r| \ge t$ holds for all $F_1, \ldots, F_r \in \mathscr{F}$. Let us define r-wise t-intersecting families $\mathscr{F}_i(n,k,r,t)$ as follows:

$$\mathscr{F}_i(n,k,r,t) = \{F \in \binom{[n]}{k} : |F \cap [t+ri]| \ge t + (r-1)i\}.$$

Let $m(n,k,r,t)$ be the maximal size of k-uniform r-wise t-intersecting families on n vertices. Can we extend the Erdős–Ko–Rado Theorem in the following way?

Conjecture 1. $m(n,k,r,t) = \max_i |\mathscr{F}_i(n,k,r,t)|$.

Ahlswede and Khachatrian[1] proved the case $r = 2$, which extended the earlier results by Erdős–Ko–Rado[3], Frankl[6] and Wilson[25]. Frankl proved the case $t = 1$ as follows.

Theorem 1 ([4])**.** $m(n,k,r,1) = \binom{n-1}{k-1}$ *for* $(r-1)n \ge rk$.

The cases $r \ge 3$ and $t \ge 2$ seem to be much more difficult and only a few results are known.

Theorem 2 ([9, 10])**.** $m(n,k,3,2) = \binom{n-2}{k-2}$ *for* $\frac{k}{n} < 0.501$ *and* $n > n_0$.

Theorem 3 ([23])**.** $m(n,k,3,t) = \binom{n-t}{k-t}$ *for* $t \ge 26$, $\frac{k}{n} \le \frac{2}{\sqrt{4t+9}-1}$ *and* $n > n_0(t)$.

Theorem 4 ([22])**.** $m(n,k,r,t) = \binom{n-t}{k-t}$ *if* $p = \frac{k}{n}$ *satisfies* $p < \frac{r-2}{r}$,

$$(1-p)p^{\frac{t}{t+1}(r-1)} - p^{\frac{t}{t+1}} + p < 0$$

and $n > n_0(r,t,p)$.

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Our main result in this paper is the following.

Theorem 5. *Let t be an integer with $1 \le t \le 7$. Then there exists $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$ such that $m(n,k,4,t) = \binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0$. Moreover $\mathscr{F}_0(n,k,4,t)$ is the only optimal configuration (up to isomorphism).*

There is a possibility to improve the range for t in the above theorem from $t \le 7$ to $t \le 10$, but the theorem fails for $t \ge 11$. In fact, by simple computation, one finds $|\mathscr{F}_1(n,k,4,t)| > |F_0(n,k,4,t)|$ if $\frac{k}{n} > \frac{1}{2}$ and $t = 11$, or $\frac{k}{n} \ge \frac{1}{2}$ and $t \ge 12$.

A family $\mathscr{F} \subset 2^{[n]}$ is called r-wise t-union if $|F_1 \cup \cdots \cup F_r| \le n-t$ holds for all $F_1, \dots, F_r \in \mathscr{F}$. This is equivalent to the property that $\mathscr{F}^c = \{[n] - F : F \in \mathscr{F}\}$ is r-wise t-intersecting. What is the maximal size of r-wise t-intersecting and q-wise t-union k-uniform family? The case $r \ge 4$, $q \ge 4$ and $t = 1$ was settled as follows.

Theorem 6 ([16, 2])**.** *Let $r \ge 4$, $q \ge 4$ and $\mathscr{F} \subset \binom{[n]}{k}$. Suppose that $\mathscr{F}$ is r-wise 1-intersecting and q-wise 1-union, and*

$$\frac{n-1}{q} + 1 \le k \le \frac{r-1}{r}(n-1).$$

Then we have $|\mathscr{F}| \le \binom{n-2}{k-1}$.

The case $r = q = 3$ and $t = 1$ is more difficult and still open. As a special case the following is known.

Theorem 7 ([11])**.** *Let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 3-wise 1-intersecting and 3-wise 1-union family. Then we have $|\mathscr{F}| \le \binom{2n-2}{n-1}$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-1]}{n} : 1 \in F\}$.*

In [21] the case $r = q = 4$ and $t = 2$ was considered. Using Theorem 5 we extend the result as follows.

Theorem 8. *Let t be an integer with $1 \le t \le 4$, and let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 4-wise t-intersecting and 4-wise t-union family. Then we have $|\mathscr{F}| \le \binom{2n-2t}{n-t}$ for $n > n_0$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-t]}{n} : [t] \subset F\}$.*

A family $\mathscr{F} \subset 2^{[n]}$ is called a Sperner family if $F \not\subset G$ holds for all distinct $F, G \in \mathscr{F}$. What is the maximum size of r-wise t-intersecting families? The case $r = 2$ was determined by Milner in [19], and the maximum is given by the simple formula $\binom{n}{\lceil (n+t)/2 \rceil}$. For the cases $r \ge 3$, the situation becomes more complicated. Frankl[4] and Gronau[12, 13, 14, 15] considered the case $r = 3$ and $t = 1$, and it is known that for $n \ge 53$ the only optimal families are

$$\begin{aligned} \mathscr{F} &= \{F \cup \{n\} : F \in \tbinom{[n-1]}{n/2}\} \cup \{[n-1]\} && n \text{ even}, \\ \mathscr{F} &= \{F \cup \{n\} : F \in \tbinom{[n-1]}{(n-1)/2}\} && n \text{ odd}. \end{aligned}$$

The case $r = 3$ and $t = 2$ was solved in [9, 10] as follows.

Theorem 9. *Let $\mathscr{F} \subset 2^{[n]}$ be a 3-wise 2-intersecting Sperner family. Then,*

$$|\mathscr{F}| \leq \begin{cases} \binom{n-2}{(n-2)/2} & \text{if } n \text{ even}, \\ \binom{n-2}{(n-1)/2} + 2 & \text{if } n \text{ odd}, \end{cases}$$

holds for $n \geq n_0$. The extremal configurations are

$$\mathscr{F} = \{\{1,2\} \cup F : F \in \binom{[3,n]}{(n-2)/2}\} \qquad n \text{ even},$$
$$\mathscr{F} = \{\{1,2\} \cup F : F \in \binom{[3,n]}{(n-1)/2}\} \cup \{[n] - \{1\}\} \cup \{[n] - \{2\}\} \qquad n \text{ odd}.$$

In this paper we consider the case $r = 4$ and $1 \leq t \leq 7$ and we prove the following.

Theorem 10. *Let $1 \leq t \leq 7$ and let $\mathscr{F} \subset 2^{[n]}$ be a 4-wise t-intersecting Sperner family. Then we have $|\mathscr{F}| \leq \binom{n-t}{\lceil \frac{n-t}{2} \rceil}$ for $n > n_0$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[n]}{k} : [t] \subset F\}$ where $k = t + \lceil \frac{n-t}{2} \rceil$ or $k = t + \lfloor \frac{n-t}{2} \rfloor$.*

We present the proofs of Theorem 5, Theorem 8 and Theorem 10 in Section 3, Section 4 and Section 5, respectively. In the next section we review some basic tools for those proofs.

2. Tools

For integers $1 \leq i < j \leq n$ and a family $\mathscr{F} \subset \binom{[n]}{k}$, define the (i,j)-shift S_{ij} as follows.

$$S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\},$$

where

$$S_{ij}(F) = \begin{cases} (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ (F - \{j\}) \cup \{i\} \notin \mathscr{F}, \\ F & \text{otherwise}. \end{cases}$$

A family $\mathscr{F} \subset \binom{[n]}{k}$ is called shifted if $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $1 \leq i < j \leq n$. For a given family $\mathscr{F}$, one can always obtain a shifted family $\mathscr{F}'$ from $\mathscr{F}$ by applying shifting to $\mathscr{F}$ repeatedly. Then we have $|\mathscr{F}'| = |\mathscr{F}|$ because shifting preserves the size of the family. It is easy to check that if $\mathscr{F}$ is r-wise t-intersecting then $S_{ij}(\mathscr{F})$ is also r-wise t-intersecting. Therefore if $\mathscr{F}$ is an r-wise t-intersecting family then we can find a shifted family $\mathscr{F}'$ which is also r-wise t-intersecting with $|\mathscr{F}'| = |\mathscr{F}|$. See [7] for more details.

We use the random walk method originated from [5, 6] by Frankl. Let us introduce a partial order in $\binom{[n]}{k}$ by using shifting. For $F, G \in \binom{[n]}{k}$, define $F \succ G$ if G is obtained by repeating a shifting to F. The following fact follows immediately from the definition.

Fact 1. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted family. If $F \in \mathscr{F}$ and $F \succ G$, then $G \in \mathscr{F}$.*

For $F \in \binom{[n]}{k}$ we define the corresponding walk on $\mathbb{Z}^2$, denoted by walk(F), in the following way. The walk is from $(0,0)$ to $(n-k,k)$ with n steps, and if $i \in F$ (resp. $i \notin F$) then the i-th step is one unit up (resp. one unit to the right). The following fact is useful (see [5, 7, 21]).

Fact 2. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family. Then for all $F \in \mathscr{F}$, walk(F) must touch the line $L : y = (r-1)x + t$.*

The next result (Corollary 8 in [21]) enables us to upper bound the number of walks which touch a given line.

Proposition 11. *Let $p \in \mathbb{Q}$, $r,s,u,v \in \mathbb{N}$ be fixed constants with $r \geq 2$ and $p < \frac{r-1}{r+1}$, and let n and k be positive integers with $p = \frac{k}{n}$. Let $\alpha \in (p,1)$ be the unique root of the equation $(1-p)x^r - x + p = 0$ and let $g(n)$ be the number of walks from (u,v) to $(n-k,k)$ which touch the line $y = (r-1)(x-u)+v+s$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{g(n)}{\binom{n-u-v}{k-v}} \leq (1+\varepsilon)\alpha^s$$

holds for all $n > n_0$. Moreover if $u = 0$ then we can choose $\varepsilon = 0$.

To prove Theorem 8 we use a dual version of Fact 2.

Fact 3. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be a shifted q-wise s-union family. Then for all $F \in \mathscr{F}$, walk(F) must touch the line $L_2 : y = \frac{1}{q-1}(x-n+k+s)+k$.*

Then we can extend Proposition 11 as follows (Corollary 9 in [21]).

Proposition 12. *Let $q,r,s,t,u,v \in \mathbb{N}$ be fixed constants with $q \geq 4$, $r \geq 4$ and $t+(r-1)u-v > 0$. Let $\alpha_j \in (\frac{1}{2},1)$ be the unique root of the equation $\frac{1}{2}x^j - x + \frac{1}{2} = 0$. Let $h(n)$ be the number of walks from (u,v) to (n,n) which touch both of the lines $L_1 : y = (r-1)x+t$ and $L_2 : y = \frac{1}{q-1}(x-n+s)+n$. Then for any $\varepsilon > 0$ there exists n_0 such that*

$$\frac{h(n)}{\binom{2n-u-v}{n-v}} \leq (1+\varepsilon)\alpha_r^{t+(r-1)u-v}\alpha_q^s$$

holds for all $n > n_0$.

To prove Theorem 10, we need a basic fact about shadow. For a family $\mathscr{F} \subset 2^{[n]}$ and a positive integer $\ell < n$, let us define the ℓ-th shadow of $\mathscr{F}$, denoted by $\Delta_\ell(\mathscr{F})$, as follows.

$$\Delta_\ell(\mathscr{F}) = \{G \in \binom{[n]}{\ell} : G \subset \exists F \in \mathscr{F}\}.$$

We use the following version of the Kruskal–Katona Theorem [18, 17, 8].

Proposition 13. *Suppose that $\mathscr{F} \subset \binom{[n]}{k}$ and $|\mathscr{F}| \leq \binom{m}{k}$. Then we have*

$$|\Delta_\ell(\mathscr{F})| \geq |\mathscr{F}|\binom{m}{\ell}/\binom{m}{k}.$$

Equality holds only if $\mathscr{F} = \binom{Y}{k}$, $|Y| = m$.

3. Multiply intersecting families

In this section we prove Theorem 5. Note that $|\mathscr{F}_0(n,k,r,t)| = \binom{n-t}{k-t} \approx p^t\binom{n}{k}$, and $|\mathscr{F}_1(n,k,r,t)| = (t+r)\binom{n-t-r}{k-t-r+1} + \binom{n-t-r}{k-t-r} \approx \big((t+r)p^{t+r-1} - (t+r-1)p^{t+r}\big)\binom{n}{k}$, where we denote $a \approx b$ iff $\lim_{n\to\infty} a/b = 1$. Let $p_{r,t} \in (0,1)$ be the root of the equation $1 = (t+r)x^{r-1} - (t+r-1)x^r$. Then $|\mathscr{F}_0(n,k,r,t)| > |\mathscr{F}_1(n,k,r,t)|$ holds if $p \le p_{r,t}$. Throughout this section, we assume that $0 < p \le p_{r,t}$ and let $q = 1-p$. We start with the following somewhat cumbersome statement, which will imply Theorem 5 as a special case after some refinement (see Proposition 15).

Proposition 14. *Let $r,t \in \mathbb{N}$ and $p \in \mathbb{Q}$ be given. Suppose that $r \ge 3$ and $p \in (0, 0.55)$. Let $\alpha \in (p,1)$ be the root of the equation $qx^r - x + p = 0$. Suppose that r,t,p satisfy all of the following inequalities:*

$$(C1) \qquad (\alpha/p)^t - t(1-\alpha^{r-1})p^{r-1}q^2 + \alpha^{r-1}q + p - 2 < 0,$$

$$(C2) \qquad (\alpha/p)^t - 1 - \frac{1-\alpha^{r-1}}{\alpha^{2r-2}}q\big(1-(p/\alpha)\big) < 0,$$

$$(C3) \qquad \frac{\alpha^{2(r-1)}}{t(1-\alpha^{r-1})q}\sum_{j=0}^{t+r-2}(j+1)(\alpha/p)^{t+r-1-j} - 1 < 0.$$

Then $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $p = \frac{k}{n}$ and $n > n_0(r,t,p)$. Moreover $\mathscr{F}_0(n,k,r,t)$ is the only optimal configuration (up to isomorphism).

We prove Proposition 14 in section 3.1 and we will show that we can replace $(C1)$ by weaker conditions in section 3.2 (see Proposition 15). Then Theorem 5 will follow from Proposition 15 easily.

3.1. **Proof of Proposition 14.**

Let $p \in \mathbb{Q}$ with $0 < p \le 0.55$ be given. Let $\alpha = \alpha_p \in (p,1)$ be the root of the equation $qx^r - x + p = 0$.

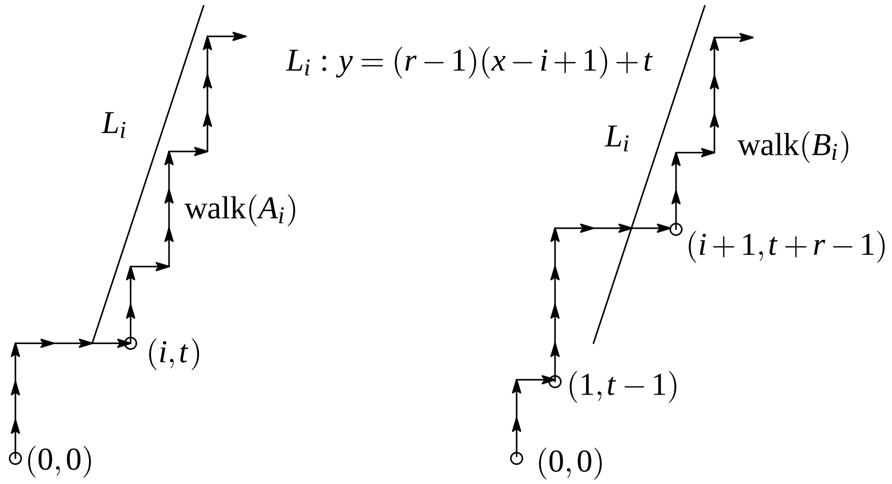
Let $\mathscr{H} \subset \binom{[n]}{k}$ be a shifted r-wise t-intersecting family and suppose that $p = \frac{k}{n}$. Then by Fact 2 walk(H) hits the line $L: y = (r-1)x + t$ for all $H \in \mathscr{H}$. Thus by Proposition 11 (setting $u = v = 0$, $s = t$) we have $|\mathscr{H}| \le \alpha^t\binom{n}{k}$. Our goal is to prove that $|\mathscr{H}| < \binom{n-t}{k-t} \approx p^t\binom{n}{k}$ unless $\mathscr{H} \cong \mathscr{F}_0(n,k,r,t)$.

For $0 \le i \le \lfloor\frac{k-t}{r-1}\rfloor$ let us define

$$\mathscr{G}_i = \{G \in \binom{[n]}{k} : |G \cap [t+r\ell]| \ge t + (r-1)\ell \text{ first holds at } \ell = i\}.$$

In other words, $G \in \mathscr{G}_i$ iff walk(G) reaches the line L at $(i, (r-1)i+t)$ for the first time. Set $\mathscr{H}_i = \mathscr{H} \cap \mathscr{G}_i$.

Next we will define $A_i \in \mathscr{G}_0$ and $B_i \in \mathscr{G}_1$. As in the following picture, starting from the origin, walk(A_i) passes $(0,t)$ and (i,t), and then from (i,t) walk(A_i) is the maximal walk (in the shifting poset) that does not touch the line $L_i: y = (r-1)(x-i) + (t+r-1)$, while walk($B_i$) passes $(0,t-1)$, $(1,t-1)$, $(1,t+r-1)$, and $(i+1,t+r-1)$, then from $(i+1,t+r-1)$ walk(B_i) is the maximal walk that does not touch the line L_i.



Formal definitions are as follows. For an infinite set $A = \{a_1, a_2, \dots\} \subset \mathbb{N}$ with $a_1 < a_2 < \cdots$, let us define $\mathrm{First}_k(A) = \{a_1, a_2, \dots, a_k\}$. Set

$$\begin{aligned} T(i) &= \{i, i+r, i+2r, \dots\} = \{i+rj : j \geq 0\}, \\ A_i^* &= [t] \cup \left(\bigcup \{T(t+i+s) : 1 \leq s \leq r, s \neq r-1\}\right) \\ &= \big([t] \cup [t+i+1, \infty]\big) - \bigcup_{j=0}^{\infty} \{t+i+r-1+rj\}, \\ B_i^* &= [t-1] \cup [t+1, t+r] \cup \left(\bigcup \{T(t+i+s+r) : 1 \leq s \leq r, s \neq r-1\}\right) \\ &= \big([t-1] \cup [t+1, t+r] \cup [t+i+r+1, \infty)\big) - \bigcup_{j=1}^{\infty} \{t+i+r-1+rj\} \end{aligned}$$

and define $A_i = \mathrm{First}_k(A_i^*)$, $B_i = \mathrm{First}_k(B_i^*)$. We will use only small i so that $A_i, B_i \in \binom{[n]}{k}$, and then we have $A_i \in \mathscr{G}_0$ and $B_i \in \mathscr{G}_1$. Note that $A_{i+1} \succ A_i$ and $B_{i+1} \succ B_i$.

We consider three cases according to the structure of $\mathscr{H}$. If $\mathscr{H}$ is similar to $\mathscr{F}_0(n,k,r,t)$ then we compare $\mathscr{H}$ with $\mathscr{F}_0(n,k,r,t)$ and this is Case 2. In Case 3 we compare $\mathscr{H}$ with $\mathscr{F}_1(n,k,r,t)$. If $\mathscr{H}$ is neither similar to $\mathscr{F}_0$ nor $\mathscr{F}_1$ then it is less likely that $\mathscr{H}$ has large size, but in this case we do not have an appropriate comparison object, which makes it difficult to bound the size of $\mathscr{H}$. We deal with this situation in Case 1, and we will refine the estimation for this case in the next subsection again.

Case 1. $A_1 \notin \mathscr{H}$ and $B_1 \notin \mathscr{H}$.
Suppose that $H \in \mathscr{H}_0$. Then after passing the point $(0,t)$, walk(H) goes to $(0,t+1)$ or $(1,t)$. So we can divide $\mathscr{H}_0 = \mathscr{H}_0^{(0,t+1)} \cup \mathscr{H}_0^{(1,t)}$ according to the next point to $(0,t)$ in

the walk. For $\mathscr{H}_0^{(0,t+1)}$ we use a trivial bound

$$|\mathscr{H}_0^{(0,t+1)}| \le \binom{n-(t+1)}{k-(t+1)} \approx p^{t+1}\binom{n}{k}. \tag{1}$$

If $H \in \mathscr{H}_0^{(1,t)}$ then walk(H) must touch the line $L : y = (r-1)x+t$ after passing $(1,t)$. Otherwise we get $H \succ A_1$, which means $H \notin \mathscr{H}$ by Fact 1, a contradiction. Here we used the fact that A_1 is the minimal set (in the shifting order poset) whose walk does not touch the line L after passing $(1,t)$. Thus by Proposition 11 (setting $u=1$, $v=t$, $s=r-1$) we have

$$|\mathscr{H}_0^{(1,t)}| \le (1+\varepsilon)\alpha^{r-1}\binom{n-(t+1)}{k-t} \approx \alpha^{r-1}p^t q\binom{n}{k}. \tag{2}$$

Next suppose that $H \in \mathscr{H}_1$. Then after passing $(1,t+r-1)$, walk(H) goes to $(1,t+r)$ or $(2,t+r-1)$. So we can divide $\mathscr{H}_1 = \mathscr{H}_1^{(1,t+r)} \cup \mathscr{H}_1^{(2,t+r-1)}$. Noting that there are t ways of walking from $(0,0)$ to $(1,t+r)$ which avoid passing $(0,t)$, we have

$$|\mathscr{H}_1^{(1,t+r)}| \le t\binom{n-(t+r+1)}{k-(t+r)} \approx tp^{t+r}q\binom{n}{k}. \tag{3}$$

If $H \in \mathscr{H}_1^{(2,t+r-1)}$, then walk$(H)$ must touch L after passing $(2,t+r-1)$. Otherwise we get $H \succ B_1$, which means $H \notin \mathscr{H}$, a contradiction. Thus by Proposition 11 (setting $u=2$, $v=t+r-1$, $s=r-1$) we have

$$|\mathscr{H}_1^{(2,t+r-1)}| \le (1+\varepsilon)t\alpha^{r-1}\binom{n-(t+r+1)}{k-(t+r-1)} \approx t\alpha^{r-1}p^{t+r-1}q^2\binom{n}{k}. \tag{4}$$

Finally we count the number of H in $\bigcup_{i\ge 2}\mathscr{H}_i \subset \bigcup_{i\ge 2}\mathscr{G}_i$. By Proposition 11 (setting $u=v=0$, $s=r$) we have $|\bigcup_{i\ge 0}\mathscr{G}_i| \le \alpha^t\binom{n}{k}$ and so

$$\begin{aligned}|\bigcup_{i\ge 2}\mathscr{H}_i| &\le |\bigcup_{i\ge 0}\mathscr{G}_i| - |\mathscr{G}_0| - |\mathscr{G}_1| \\ &\le \alpha^t\binom{n}{k} - \binom{n-t}{k-t} - t\binom{n-(t+r)}{k-(t+r-1)} \\ &\approx (\alpha^t - p^t - tp^{t+r-1}q)\binom{n}{k}. \end{aligned} \tag{5}$$

Therefore by (1), (2), (3), (4) and (5) we have

$$\frac{|\mathscr{H}|}{\binom{n}{k}} \le (1+o(1))(p^{t+1}+\alpha^{r-1}p^tq+tp^{t+r}q+t\alpha^{r-1}p^{t+r-1}q^2+\alpha^t-p^t-tp^{t+r-1}q)$$

as $n\to\infty$. Consequently $|\mathscr{H}| < \binom{n-t}{k-t} \approx p^t\binom{n}{k}$ follows from

$$p^{t+1}+\alpha^{r-1}p^tq+tp^{t+r}q+t\alpha^{r-1}p^{t+r-1}q^2+\alpha^t-p^t-tp^{t+r-1}q < p^t,$$

which is equivalent to $(C1)$.

Case 2. $A_1 \in \mathscr{H}$.
If $[t] \subset H$ holds for all $H \in \mathscr{H}$ then it follows that $|\mathscr{H}| \leq \binom{n-t}{k-t}$ and equality holds iff $\mathscr{H} \cong \mathscr{F}_0(n,k,r,t)$. Thus we may assume that $[t] \not\subset H$ holds for some $H \in \mathscr{H}$ and in particular we may assume that $D' = [k+1] - \{t\} \in \mathscr{H}$ because $\mathscr{H}$ is shifted.

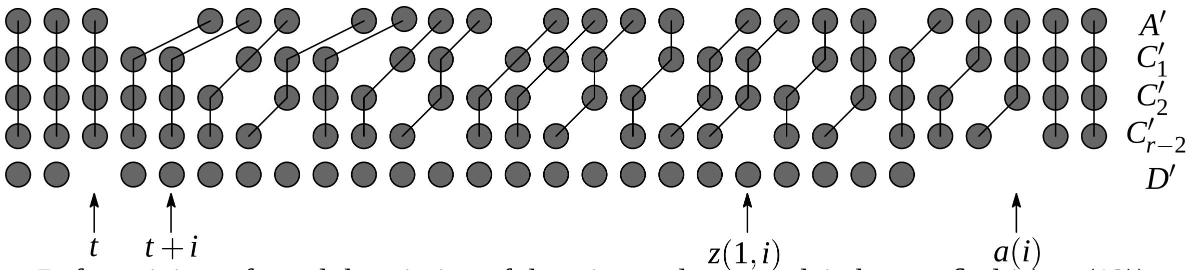
We shall show that $A_i \notin \mathscr{H}$ holds for some i. Our plan is to choose a "witness" $\{A', C'_1, \ldots, C'_{r-2}\}$ for being $A_i \notin \mathscr{H}$ so that

$$A_i \succ A' \succ C'_1 \succ C'_2 \succ \cdots \succ C'_{r-2}, \tag{6}$$

and

$$A' \cap C'_1 \cap C'_2 \cap \cdots \cap C'_{r-2} \cap D' = [t-1]. \tag{7}$$

Suppose that we have chosen the witness. If $A_i \in \mathscr{H}$ then (6) and Fact 1 imply $A', C'_1, \ldots, C'_{r-2} \in \mathscr{H}$, and thus (7) contradicts that $\mathscr{H}$ is r-wise t-intersecting. The following picture shows an example of a witness for the case $r = 5, t = 3, i = 2$ and $k = 23$. Lines connecting the discs show that $A' \succ C'_1 \succ C'_2 \succ C'_3$.



Before giving a formal description of the witness, let us explain how to find i (see (13)) by considering a bit more rough situation. Here we consider infinite sets for simplicity. Let

$$A'' = [t] \cup [t+i+1, \infty) - \{t+i+rj+r-1 : j \geq 0\}.$$

We try to find $C''_1, \ldots, C''_{r-2}$ so that

$$A'' \succ C''_1 \succ C''_2 \succ \cdots \succ C''_{r-2}, \tag{8}$$

$$A'' \cap C''_1 \cap C''_2 \cap \cdots \cap C''_{r-2} = [t]. \tag{9}$$

To do so, we maintain

$$|A'' \cap \{j\}| + |C''_1 \cap \{j\}| + \cdots + |C''_{r-2} \cap \{j\}| = r-2 \tag{10}$$

for all $j > t+i$ by using a cyclic pattern. More formally, set $z(u,i) = t+i+u(r-2)r$, and for $1 \leq \ell \leq r-2$ set $C''_\ell = [1,\infty) - Z_\ell(i)$, where

$$Z_1(i) = \bigcup_{u \geq 0} (\{z(u,i)+1, z(u,i)+r\} \cup \{z(u,i)+(r-1)v : 2 \leq v \leq r-2\}),$$

and $Z_\ell(i) = \{t+i+\ell\} \cup (r + Z_{\ell-1}(i))$ for $2 \leq \ell \leq r-2$. Here we denote the set $\{r+z : z \in Z\}$ by $r+Z$. In $[t+i+1,\infty)$, the sets $A'', C''_1, \ldots, C''_{r-2}$ are periodic of period $r(r-2)$. Due to (10), we have (9). But (8) is not satisfied. So we will find an integer a such that

$$\mathrm{First}_a(A'') \succ \mathrm{First}_a(C''_1) \succ \mathrm{First}_a(C''_2) \succ \cdots \succ \mathrm{First}_a(C''_{r-2}), \tag{11}$$

It is necessary that

$$|A'' \cap [a]| = |C''_\ell \cap [a]| \tag{12}$$

holds for all $1 \le \ell \le r-2$. We need to adjust the excess $|C''_\ell \cap [t+i]| - |A'' \cap [t+i]| = i$. We note that

$$A'' \cap [t+i+1, t+i+r(r-2)] = (r-1)(r-2),$$
$$C''_\ell \cap [t+i+1, t+i+r(r-2)] = (r-1)(r-2)-1,$$

and

$$A'' \cap [t+i+1, t+i+(2r-3)] = 2r-4,$$
$$C''_\ell \cap [t+i+1, t+i+(2r-3)] = 2r-5.$$

Thus we find that

$$a = t+i+(i-1)r(r-2)+(2r-3) = t+(r-1)((r-1)i-r+3)$$

satisfies (12). We leave the reader to check that a defined above satisfy (11), actually this is the maximum integer satisfying (11). We require $a \ge k+1$, which gives $i \ge i_0$ where

$$i_0 = \left\lceil \frac{k+1-t+(r-1)(r-3)}{(r-1)^2} \right\rceil. \tag{13}$$

Now we are ready to define the witness $A', C'_1, \dots, C'_{r-2}$. Set

$$\begin{aligned} \tilde{A} &= [t] \cup ([t+i_0+1, a(i_0)] - \{t+i_0+rj+r-1 : j \ge 0\}) \cup [a(i_0)+1, \infty) \\ &= (A_{i_0} \cap [a(i_0)]) \cup [a(i_0)+1, \infty) \end{aligned}$$

where $a(i) = t-(r-1)(r-3)+(r-1)^2 i$ and define $A' = \mathrm{First}_k(\tilde{A})$. Set

$$\tilde{C}_\ell = ([a(i_0)] - Z_\ell(i_0)) \cup [a(i_0)+1, \infty)$$

and define $C'_\ell = \mathrm{First}_k(\tilde{C}_\ell)$ for $1 \le \ell \le r-2$. Then the witness satisfies (6) and (7). Thus we have $A' \notin \mathscr{H}$, and since $A_i \succ A'$ for $i \ge i_0$ we also have $A_i \notin \mathscr{H}$ if $i \ge i_0$.

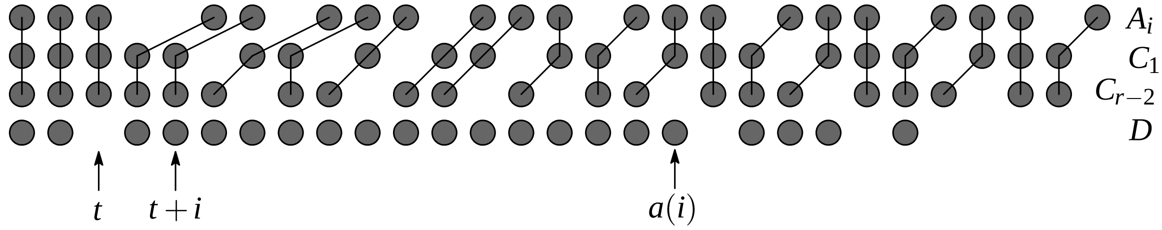
Now let $1 \le i < i_0$ be such that $A_i \in \mathscr{H}$ but $A_{i+1} \notin \mathscr{H}$. (Then $A_j \in \mathscr{H}$ iff $j \le i$.) For $1 \le \ell \le r-2$ set $R_\ell(i) = (A_i + \ell) - [a(i)]$ and

$$C^*_\ell = ([a(i)] - Z_\ell(i)) \cup R_\ell(i)$$

and let

$$D^* = ([a(i)] - \{t\}) \cup R_{r-1}(i).$$

Finally set $C_\ell = \mathrm{First}_k(C^*_\ell)$, $D = \mathrm{First}_k(D^*)$. The following picture shows an example of the case $r=4, t=3, i=2$ and $k=21$.

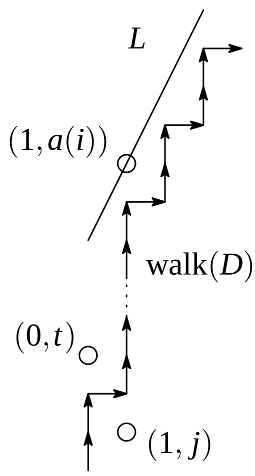


Then we have $C_\ell \in \mathscr{H}$ because $A_i \in \mathscr{H}$ and $A_i \succ C_1 \succ C_2 \succ \cdots \succ C_{r-2}$. Since $\mathscr{H}$ is r-wise t-intersecting and $A_i \cap C_1 \cap C_2 \cap \cdots \cap C_{r-2} \cap D = [t-1]$ we can conclude that $D \notin \mathscr{H}$. Since $a(i) - (r-2) \equiv t + i - 1 \pmod r$ we have $A_i \not\ni a(i) - (r-2)$, and thus $R_{r-1}(i) \cup [a(i)] = A_i + (r-1) \not\ni a(i) + 1$. This means that after passing $(0, t-1)$ and $(1, t-1)$, walk(D) is the maximal walk that does not touch the line $L : y = (r-1)(x-1) + a(i)$.

Let $H \in \mathscr{H}$. First suppose that walk(H) does not pass $(0,t)$, i.e., $H \cap [t] \neq [t]$. Then walk(H) must go through at least one of the points in

$$P = \{(1,0), (1,1), \ldots, (1,t-1)\}.$$

Let $(1, j)$ $(0 \leq j \leq t-1)$ be the first point in P that walk(H) hits. In other words, we have $H \cap [j+1] = [j]$. From the point $(1, j)$, walk(H) must touch the line L, otherwise we get $H \succ D$ and $D \in \mathscr{H}$, which is a contradiction.



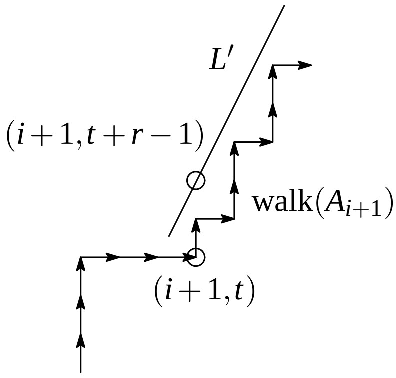
We estimate the number of walks from $(1, j)$ to $(n-k, k)$ which touch the line L. By Proposition 11 (setting $u = 1$, $v = j$, $s = a(i) - j$) the number is at most

$$(1+\varepsilon)\alpha^{a(i)-j}\binom{n-(j+1)}{k-j}.$$

Therefore the number of $H \in \mathscr{H}$ such that $H \cap [t] \neq [t]$ is at most

$$(1+\varepsilon)\sum_{j=0}^{t-1}\alpha^{a(i)-j}\binom{n-(j+1)}{k-j}. \tag{14}$$

Next suppose that walk(H) passes $(0,t)$, i.e., $H \cap [t] = [t]$. The number of corresponding walks is at most $\binom{n-t}{k-t}$, but we need to refine this estimation. Suppose that walk(H) passes $(i+1, t)$. Then from this point walk(H) must touch the line $L' : y = (r-1)(x-(i+1)) + t + r - 1$, otherwise we get $H \succ A_{i+1}$ and $A_{i+1} \in \mathscr{H}$, which is a contradiction.



The trivial upper bound for the number of walks from $(i+1,t)$ to $(n-k,k)$ is $\binom{n-(t+i+1)}{k-t}$, but those walks in $\mathscr{H}$ touch the line L' and so by Proposition 11 we will get an improved upper bound. To apply the proposition, it is convenient to neglect the first $i+t+1$ steps of the walks, in other words, we shift the origin to $(i+1,t)$, and replace n and k by $n' = n-(t+i+1)$ and $k' = k-t$. Then L' becomes $y = (r-1)x + r - 1$ in the new coordinates, and by setting $u = v = 0$ and $s = r-1$, Proposition 11 gives an improved upper bound $\alpha_{p'}^{r-1}\binom{n'}{k'}$ where $p' = \frac{k'}{n'} \approx \frac{k}{n-i}$ and $\alpha_{p'} \in (p',1)$ be the root of the equation $(1-p')x^r - x + p' = 0$. Therefore the number of $H \in \mathscr{H}$ such that $H \cap [t] = [t]$ is at most

$$\binom{n-t}{k-t} - (1-\alpha_{p'}^{r-1})\binom{n'}{k'}. \tag{15}$$

We shall show $|\mathscr{H}| < \binom{n-t}{k-t}$. By (14) and (15) it suffices to prove that

$$(1+\varepsilon)\sum_{j=0}^{t-1}\alpha^{a(i)-j}\binom{n-(j+1)}{k-j} - (1-\alpha_{p'}^{r-1})\binom{n'}{k'} < 0,$$

or equivalently,

$$(1+\varepsilon)\sum_{j=0}^{t-1}\alpha^{t-(r-1)(r-3)-j}\binom{n-(j+1)}{k-j} < \frac{1-\alpha_{p'}^{r-1}}{\alpha^{(r-1)^2 i}}\binom{n'}{k'} =: f(i). \tag{16}$$

Claim 1. *$f(i)$ is an increasing function of i.*

Proof. To show $f(i-1) < f(i)$, let $p'' = \frac{k-t}{n-(t+(i-1)+1)} = \frac{k'}{n'+1}$. Then we need to show

$$\frac{1-\alpha_{p''}^{r-1}}{\alpha^{(r-1)^2(i-1)}}\binom{n'+1}{k'} < \frac{1-\alpha_{p'}^{r-1}}{\alpha^{(r-1)^2 i}}\binom{n'}{k'},$$

which is equivalent to

$$\frac{1-\alpha_{p''}^{r-1}}{1-\alpha_{p'}^{r-1}} < \frac{1}{\alpha^{(r-1)^2}}\binom{n'}{k'}\Big/\binom{n'+1}{k'} = \frac{1}{\alpha^{(r-1)^2}}\cdot\frac{n'+1-k'}{n'+1}.$$

Using (13) we have $\frac{n'+1-k}{n'+1} = \frac{n-k-i}{n-t-i} \geq \frac{n-k-i_0}{n-t-i_0} \approx (1-p-\frac{p}{(r-1)^2})/(1-\frac{p}{(r-1)^2}) > (p+p^r)^{(r-1)^2} > \alpha^{(r-1)^2}$ for $p < 0.55$ and $r \geq 3$. Thus we can choose $\delta > 0$ so small that

$$1+\delta < \frac{1}{\alpha^{(r-1)^2}} \cdot \frac{n'+1-k'}{n'+1}$$

holds for $n > n_0(\delta)$. On the other hand, since $\frac{1}{p''} = \frac{1}{p'} + \frac{1}{k'}$ we have $p'' \approx p'$ and hence

$$\frac{1-\alpha_{p''}^{r-1}}{1-\alpha_{p'}^{r-1}} < 1+\delta$$

for $n > n_1(\delta)$. □

Thus it suffices to show the inequality (16) for $i=1$. Noting that $p' \approx p$, $\binom{n-(j+1)}{k-j} \approx p^j q \binom{n}{k}$ and $\binom{n-(t+2)}{k-t} \approx p^t q^2 \binom{n}{k}$, we find that the target inequality follows from $(C2)$ by choosing $\varepsilon = \varepsilon(r,t,p)$ sufficiently small.

Case 3. $B_1 \in \mathscr{H}$.
Let $D' = [k+2] - \{t+r-1, t+r\}$. If $D' \notin \mathscr{H}$ then the shiftedness of $\mathscr{H}$ implies that $\mathscr{H} \subset \mathscr{F}_1(n,k,r,t)$ and we are done. (Recall that we have $|\mathscr{F}_1(n,k,r,t)| < |\mathscr{F}_0(n,k,r,t)| = \binom{n-t}{k-t}$ for $0 < p \leq p_{r,t}$.) Thus we may assume that $D' \in \mathscr{H}$. Let $i_0 = \lceil \frac{k+r^2-5r+5-t}{(r-1)^2} \rceil$ and set

$$\begin{aligned} \tilde{B} &= ([t+r]-\{t\}) \cup ([t+r+i_0+1, b(i_0)] - \{t+r+i_0+jr-1 : j \geq 1\}) \cup [b(i_0)+1, \infty) \\ &= (B_{i_0} \cap [b(i_0)]) \cup [b(i_0)+1, \infty) \end{aligned}$$

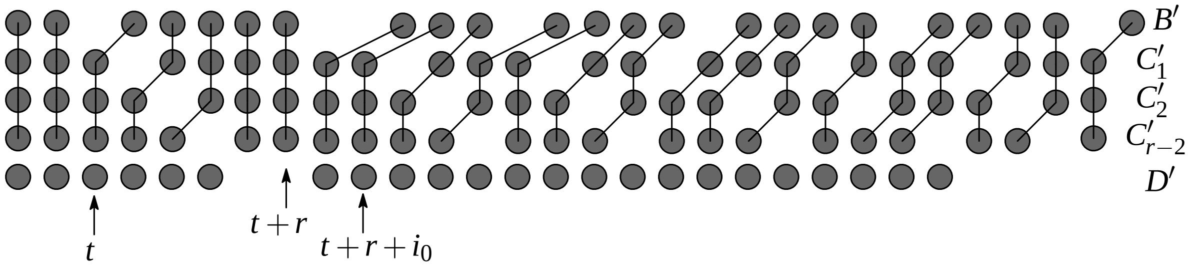
where $b(i) = t+r+i+(i-1)r(r-2)+(2r-3) = t-r^2+5r-3+(r-1)^2 i$. Set $z(u,i) = t+r+i+u(r-2)r$ and for $1 \leq \ell \leq r-1$ define $Z_\ell(i)$ by

$$Z_1(i) = \bigcup_{u \geq 0} (\{z(u,i)+1, z(u,i)+r\} \cup \{z(u,i)+(r-1)v : 2 \leq v \leq r-2\}),$$

and $Z_\ell(i) = \{t+r+i+\ell\} \cup (r+Z_{\ell-1}(i))$ for $2 \leq \ell \leq r-2$. Finally let $B' = \mathrm{First}_k(\tilde{B})$ and for $1 \leq \ell \leq r-2$ let $C'_\ell = \mathrm{First}_k(\tilde{C}_\ell)$ where

$$\tilde{C}_\ell = ([b(i_0)] - Z_\ell(i_0)) \cup [b(i_0)+1, \infty).$$

Note that $B' \succ C'_1 \succ C'_2 \succ \cdots \succ C'_{r-2}$ and $B' \cap C'_1 \cap C'_2 \cap \cdots \cap C'_{r-2} \cap D' = [t-1]$. Thus we have $B' \notin \mathscr{H}$, and since $B_i \succ B'$ for $i \geq i_0$ we also have $B_i \notin \mathscr{H}$ if $i \geq i_0$. The following picture shows an example of the case $r=5, t=3, i_0=2$ and $k=23$ ($b(i_0)=32$).



Now let $1 \le i < i_0$ be such that $B_i \in \mathscr{H}$ but $B_{i+1} \notin \mathscr{H}$. For $1 \le \ell \le r-2$ set $R_\ell(i) = (B_i + \ell) - [b(i)]$ and

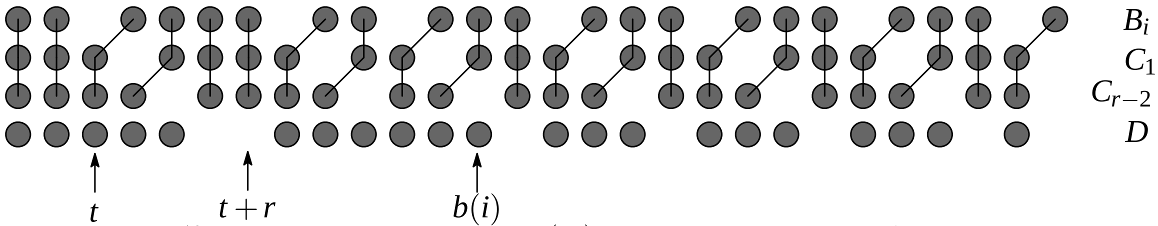
$$C_\ell^* = ([b(i)] - Z_\ell(i)) \cup R_\ell(i),$$

and let

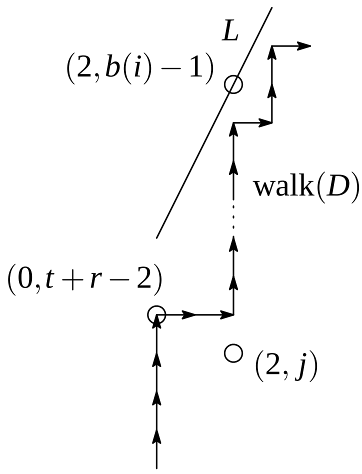
$$D^* = ([b(i)] - \{t+r-1, t+r\}) \cup R_{r-1}(i).$$

Finally set $C_\ell = \mathrm{First}_k(C_\ell^*)$, $D = \mathrm{First}_k(D^*)$.

Then we have $C_\ell \in \mathscr{H}$ because $B_i \in \mathscr{H}$ and $B_i \succ C_1 \succ C_2 \succ \cdots \succ C_{r-2}$. Since $\mathscr{H}$ is r-wise t-intersecting and $B_i \cap C_1 \cap C_2 \cap \cdots \cap C_{r-2} \cap D = [t-1]$ we can conclude that $D \notin \mathscr{H}$. The following picture shows an example of the case $r = 4, t = 3, i = 1$ and $k = 21$.



Let $H \in \mathscr{H}$. First suppose that walk(H) passes at least one of the points in $P = \{(2,0),(2,1),\ldots,(2,t+r-2)\}$, i.e., $|H \cap [t+r]| \le t+r-2$. Let $(2,j)$ $(0 \le j \le t+r-2)$ be the first point in P that walk(H) hits. From this point, walk(H) must touch the line $L: y = (r-1)(x-2) + b(i) - 1$, otherwise we get $H \succ D$ and $D \in \mathscr{H}$, a contradiction.



Thus the number of corresponding walks is at most

$$(j+1)(1+\varepsilon)\alpha^{b(i)-1-j}\binom{n-(j+2)}{k-j},$$

where $j+1$ is the number of walks from $(0,0)$ to $(2,j)$ which do not touch $\{(2,\ell) : 0 \le \ell < j\}$. Hence the number of $H \in \mathscr{H}$ such that $|H \cap [t+r]| \le t+r-2$ is at most

$$(1+\varepsilon)\sum_{j=0}^{t+r-2}(j+1)\alpha^{b(i)-1-j}\binom{n-(j+2)}{k-j}. \tag{17}$$

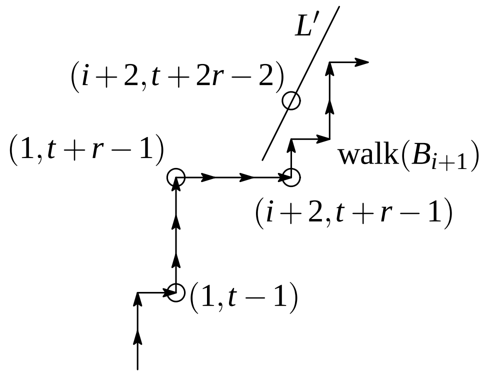
Next suppose that $|H\cap[t+r]|\ge t+r-1$. Then walk(H) passes $(0,t+r)$ or $(1,t+r-1)$. The number of walks which pass $(0,t+r)$ is at most

$$\binom{n-(t+r)}{k-(t+r)}. \tag{18}$$

The number of walks which pass $(1,t+r-1)$ is clearly at most $(t+r)\binom{n-(t+r)}{k-(t+r-1)}$ and we will improve this estimation. Suppose that walk(H) passes $(1,t-1)$, $(1,t+r-1)$ and $(i+2,t+r-1)$. Then from $(i+2,t+r-1)$, this walk must touch the line $L':y=(r-1)(x-i)+t=(r-1)(x-(i+2))+t+2r-2$, otherwise we get $H\succ B_{i+1}$ and $B_{i+1}\in\mathscr{H}$, a contradiction. Thus the number of walks in $\mathscr{H}$ which pass $(1,t+r-1)$ is at most

$$(t+r)\binom{n-(t+r)}{k-(t+r-1)}-t\big(1-\alpha_{p'}^{r-1}\big)\binom{n'}{k'}, \tag{19}$$

where $n'=n-(t+r+i+1)$, $k'=k-(t+r-1)$ and $p'=\frac{k'}{n'}\approx\frac{k}{n-i}$.



We shall show that the sum of (17), (18) and (19) is less than $|\mathscr{F}_1(n,k,r,t)|=(t+r)\binom{n-(t+r)}{k-(t+r-1)}+\binom{n-(t+r)}{k-(t+r)}$, which means $|\mathscr{H}|<|\mathscr{F}_1|$. Our target inequality is

$$(1+\varepsilon)\sum_{j=0}^{t+r-2}(j+1)\alpha^{t-(r-1)(r-4)-j}\binom{n-(j+2)}{k-j}<\frac{t(1-\alpha_{p'}^{r-1})}{\alpha^{(r-1)^2 i}}\binom{n'}{k'}.$$

One can show similarly to Claim 1 that the RHS is an increasing function of i. Thus it suffices to show the inequality for $i=1$, which follows from $(C3)$. $\square$

3.2. **Further improvement.** In the previous subsection, we proved Proposition 14. Here we will refine the proof for Case 1 to show that we can replace $(C1)$ by the following weaker conditions $(C1a)\wedge(C1b)\wedge(C1c)$:

$$(C1a)\qquad p+\alpha^{r-1}q+tp^{r-1}q^2\big(\frac{p}{q}+\alpha^{r-1}+\frac{\alpha^r}{\alpha-p}((\alpha/p)^{r-1}-1)\big)-1<0,$$

$$(C1b)\qquad (\alpha/p)^t-tp^{r-1}q^2(1+p-\alpha^{r-1})+\alpha^{r-1}q+p^2-2<0,$$

$$(C1c)\qquad p^2+\alpha^{r-1}q+tp^rq+t(p\alpha)^{r-1}q^2+\sum_{j=1}^{r-1}u_j\alpha^{rj-1}p^{r-j}q^{j+1}-1<0.$$

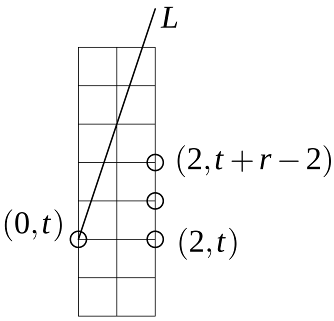
where u_j will be defined later in Case 1c.

Assume that $A_1\notin\mathscr{H}$ and $B_1\notin\mathscr{H}$. We continue to use notation defined in Case 1, and let

$$\tilde{\mathscr{H}}_0^{(0,t+1)} = \{H-[t+1]:H\in\mathscr{H}_0^{(0,t+1)}\}\subset\binom{[t+2,n]}{k-t-1},$$

$$\tilde{\mathscr{H}}_1^{(1,t+r)} = \{H\cap[t+r+2,n]:H\in\mathscr{H}_1^{(1,t+r)}\}\subset\binom{[t+r+2,n]}{k-t-r}.$$

Case 1a. $\tilde{\mathscr{H}}_0^{(0,t+1)}$ is not $(r-1)$-wise 1-intersecting.
In this case we have $G_1,\dots,G_{r-1}\in\mathscr{H}$ such that $G_1\cap\cdots\cap G_{r-1}=[t+1]$. Let $H\in\mathscr{H}$. Since $\mathscr{H}$ is r-wise t-intersecting we have $|H\cap[t+1]|\geq t$. Thus walk(H) hits $(0,t+1)$ or $(1,t)$, and walk(H) never hits a point in $\{(2,0),(2,1),\dots,(2,t-1)\}$. In particular, if $H\in\bigcup_{i\geq2}\mathscr{H}_i$ then walk(H) reaches the line $x=2$ for the first time only at one of $(2,t),\dots,(2,t+r-2)$. In this case walk(H) passes $(1,t)$ and there are t ways of walking from $(0,0)$ to $(1,t)$ which avoid $(0,t)$. Then after passing $(2,j)$ $(t\leq j\leq t+r-2)$ walk(H) must touch the line $L:y=(r-1)x+t$.



Therefore we have

$$\begin{aligned}|\bigcup_{i\geq2}\mathscr{H}_i| &\leq (1+\varepsilon)\sum_{j=t}^{t+r-2}t\alpha^{t+2r-2-j}\binom{n-(j+2)}{k-j}\\ &\approx t\alpha^rp^{t+r-2}q^2\binom{n}{k}\sum_{i=0}^{r-2}(\alpha/p)^i=t\alpha^rp^{t+r-2}q^2\frac{1-(\alpha/p)^{r-1}}{1-(\alpha/p)}\binom{n}{k}. \qquad (20)\end{aligned}$$

By (1), (2), (3), (4) and (20) it suffices to show that

$$p^{t+1}+\alpha^{r-1}p^tq+tp^{t+r}q+t\alpha^{r-1}p^{t+r-1}q^2+t\alpha^r p^{t+r-2}q^2\frac{1-(\alpha/p)^{r-1}}{1-(\alpha/p)}<p^t,$$

which is equivalent to $(C1a)$.

Case 1b. Both $\tilde{\mathscr{H}}_0^{(0,t+1)}$ and $\tilde{\mathscr{H}}_1^{(1,t+r)}$ are $(r-1)$-wise 1-intersecting.
In this case we use Theorem 1 to bound the sizes of $\mathscr{H}_0^{(0,t+1)}$ and $\mathscr{H}_1^{(1,t+r)}$. Then we have

$$|\mathscr{H}_0^{(0,t+1)}| = |\tilde{\mathscr{H}}_0^{(0,t+1)}| \le \binom{n-(t+1)-1}{k-(t+1)-1} \approx p^{t+2}\binom{n}{k}, \quad (21)$$

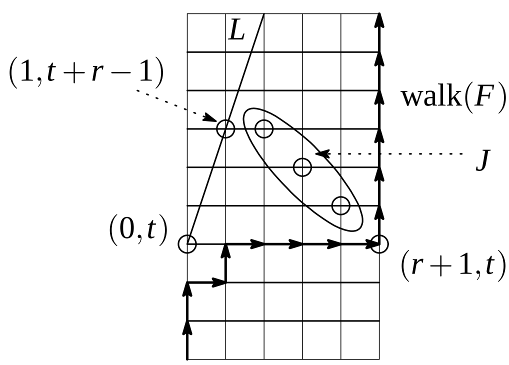
$$|\mathscr{H}_1^{(1,t+r)}| = t|\tilde{\mathscr{H}}_1^{(1,t+r)}| \le t\binom{n-(t+r+1)-1}{k-(t+r)-1} \approx tp^{t+r+1}q\binom{n}{k}. \quad (22)$$

Therefore by (21), (2), (22), (4) and (5) it suffices to show that

$$p^{t+2}+\alpha^{r-1}p^tq+tp^{t+r+1}q+t\alpha^{r-1}p^{t+r-1}q^2+\alpha^t-p^t-tp^{t+r-1}q<p^t,$$

which is equivalent to $(C1b)$.

Case 1c. $\tilde{\mathscr{H}}_0^{(0,t+1)}$ is $(r-1)$-wise 1-intersecting and $\tilde{\mathscr{H}}_1^{(1,t+r)}$ is not $(r-1)$-wise 1-intersecting.
We use (21) to bound $\mathscr{H}_0^{(0,t+1)}$ again. Now we will bound the size of $\bigcup_{i\ge 2}\mathscr{H}_i$. Since $\tilde{\mathscr{H}}_1^{(1,t+r)}$ is not $(r-1)$-wise 1-intersecting and $\mathscr{H}$ is shifted, we have $G_1,\dots,G_{r-1}\in\mathscr{H}$ such that $G_1\cap\dots\cap G_{r-1}=[t+r+1]-\{t\}$. If $F=([k+r+1]-[t,t+r+1])\cup\{t+1\}\in\mathscr{H}$ then we also have $F'=[k+r+1]-[t+1,t+r+1]\in\mathscr{H}$ by shifting. But this is impossible because $G_1\cap\dots\cap G_{r-1}\cap F'=[t-1]$. Thus we must have $F\notin\mathscr{H}$. Let $H\in\bigcup_{i\ge2}\mathscr{H}_i$. Then walk$(H)$ never hits any point in $\{(r+1,0),(r+1,1),\dots,(r+1,t)\}$, otherwise we get $H\succ F\in\mathscr{H}$, a contradiction. In other words, walk(H) passes one of the points in $J=\{(j+1,t+r-j):1\le j\le r-1\}$.



For $1\le j\le r-1$ let u_j be the number of walks from $(0,0)$ to $(j+1,t+r-j)$ which do not touch the line $L: y=(r-1)x+t$. We have $u_j=\binom{t+r+1}{j+1}-\binom{r+1}{j+1}-\delta_j$ where $\delta_1=t$

and $\delta_j = 0$ for $j \geq 2$. Then after passing $(j+1, t+r-j)$, walk(H) must touch the line L. Therefore we have

$$\begin{aligned} |\bigcup_{i\geq 2} \mathscr{H}_i| &\leq (1+\varepsilon)\sum_{j=1}^{r-1} u_j \alpha^{rj-1} \binom{n-(t+r+1)}{k-(t+r-j)} \\ &\approx \sum_{j=1}^{r-1} u_j \alpha^{rj-1} p^{t+r-j} q^{j+1} \binom{n}{k}. \end{aligned} \tag{23}$$

Consequently by (21), (2), (3), (4) and (23) it suffices to show that

$$p^{t+2} + \alpha^{r-1} p^t q + t p^{t+r} q + t\alpha^{r-1} p^{t+r-1} q^2 + \sum_{j=1}^{r-1} u_j \alpha^{rj-1} p^{t+r-j} q^{j+1} < p^t,$$

which is equivalent to $(C1c)$. □

Noting that the LHSs of $(C1a)$, $(C1b)$, $(C1c)$, $(C2)$ and $(C3)$ are continuous functions of p, we have proved the following.

Proposition 15. *Let $r, t \in \mathbb{N}$ and $p \in \mathbb{Q}$ be given. Suppose that $r \geq 3$ and $p \in (0, 0.55)$. Let $\alpha \in (0,1)$ be the root of the equation $(1-p)x^r - x + p = 0$. Suppose that r, t, p satisfy $(C1a)$, $(C1b)$, $(C1c)$, $(C2)$ and $(C3)$. Then there exists $\varepsilon = \varepsilon(r,t,p) > 0$ such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $|\frac{k}{n} - p| < \varepsilon$ and $n > n_0(r,t,p,\varepsilon)$. Moreover $\mathscr{F}_0(n,k,r,t)$ is the only optimal configuration (up to isomorphism).*

Proof of Theorem 5. Setting $r = 4$, $p = 1/2$ and $t = 1, \ldots, 7$, we can verify $(C1a)$, $(C1b)$, $(C1c)$, $(C2)$ and $(C3)$. Then the result follows from the above proposition. □

Remark 1. *In the proof of Proposition 15 and Theorem 5, we used $p \leq 0.55$ only to show*

$$(1 - p - \frac{p}{(r-1)^2})/(1 - \frac{p}{(r-1)^2}) > \alpha^{(r-1)^2}$$

for $r = 3$ (see Claim 1). If $r \geq 4$ then we can replace the condition $p \leq 0.55$ by the above inequality.

Let EKR(r) be the maximal t such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $n = 2k$ and $n > n_0$. Then EKR$(4) \geq 7$ follows from Theorem 5. Let t_r be the maximal t such that all (Ci)'s hold for $p = 1/2$ in the sense of Proposition 15, e.g., $t_4 = 7$. Clearly we have EKR$(r) \geq t_r$. On the other hand, comparing the size of $\mathscr{F}_0(n,k,r,t)$ and $\mathscr{F}_1(n,k,r,t)$, we have EKR$(r) \leq T_r = 2^r - r - 1$. If Conjecture 1 is true then it follows that EKR$(r) = T_r$. We can compute t_r and T_r for $4 \leq r \leq 10$ as follows.

r	4	5	6	7	8	9	10
t_r	7	18	41	89	184	377	762
T_r	11	26	57	120	247	502	1013

For example, $t_{10} = 762$ implies that there exists $\varepsilon > 0$ such that $m(n,k,10,t) \le \binom{n-t}{k-t}$ holds for $t \le 762$, $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0(\varepsilon)$.

Let us note that our proof of Proposition 15 also includes the proof of the following slightly stronger result.

Proposition 16. *Let $\mathscr{F} \subset \binom{[n]}{k}$ be an r-wise t-intersecting family. Suppose that $\mathscr{F}$ is non-trivial, that is, $|\bigcap_{F\in\mathscr{F}} F| < t$. Then under the same assumptions as in Proposition 15, there exist $\gamma = \gamma(r,t,p) > 0$ and $\varepsilon = \varepsilon(\gamma) > 0$ such that $|\mathscr{F}| < (1-\gamma)\binom{n-t}{k-t}$ holds for $|\frac{k}{n} - p| < \varepsilon$ and $n > n_0(\varepsilon)$.*

Let us summarize our result for the case $p = 1/2$ and $4 \le r \le 10$ as follows.

Theorem 17. *Let $4 \le r \le 10$ and let $1 \le t \le t_r$. Then there exists $\varepsilon > 0$ and $n_0 = n_0(\varepsilon)$ such that $m(n,k,r,t) = \binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0$. Moreover if $\mathscr{F}$ is non-trivial then there exist $\gamma > 0$ and $\varepsilon = \varepsilon(\gamma) > 0$ such that $|\mathscr{F}| < (1-\gamma)\binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_1(\varepsilon)$.*

4. Intersecting and union families

Proof of Theorem 8. Let $\mathscr{F} \subset \binom{[2n]}{n}$ be a 4-wise t-intersecting and 4-wise t-union family. Suppose that $\mathscr{F}$ is not 3-wise $(t+1)$-union. Then there exist $A,B,C \in \mathscr{F}$ such that $|A \cup B \cup C| = 2n - t$, say, $A \cup B \cup C = [2n-t]$. Since $\mathscr{F}$ is 4-wise t-union, we have $\mathscr{F} \subset \binom{[2n-t]}{n}$. On the other hand, $\mathscr{F}$ is 4-wise t-intersecting. Then by Theorem 5 we have $|\mathscr{F}| \le \binom{2n-2t}{n-t}$ and equality holds iff $\mathscr{F} \cong \{F \in \binom{[2n-t]}{n} : [t] \subset F\}$. This means that the theorem is true if $\mathscr{F}$ is not 3-wise $(t+1)$-union. Considering the complement, the theorem is also true if $\mathscr{F}$ is not 3-wise $(t+1)$-intersecting. Therefore from now on we assume that

$$\mathscr{F} \text{ is 3-wise } (t+1)\text{-intersecting and 3-wise } (t+1)\text{-union.} \tag{24}$$

We also assume that $\mathscr{F}$ is shifted. Now suppose that

$$|\mathscr{F}| \ge \binom{2n-2t}{n-t} \tag{25}$$

and we shall prove that there is no such $\mathscr{F}$.

Recall that for $A \in \binom{[2n]}{n}$ we define walk(A) on $\mathbb{Z}^2$ in the following way. The walk is from $(0,0)$ to (n,n) with $2n$ steps, and if $i \in A$ (resp. $i \notin A$) then the i-th step is one unit up (resp. one unit to the right). Let us define

$$\mathscr{A}_i = \{A \in \tbinom{[2n]}{n} : |A \cap [t+4\ell]| \ge t + 3\ell \text{ first holds at } \ell = i\},$$

$$\mathscr{A}_j^- = \{A \in \tbinom{[2n]}{n} : |A \cap [2n - 4\ell - t + 1, 2n]| \le \ell \text{ first holds at } \ell = j\}.$$

(Here we say that a property $P(\ell)$ first holds at $\ell = i$ iff $P(\ell)$ does not hold for $0 \le \ell < i$ and $P(i)$ holds.) If $A \in \mathscr{A}_i$ then, starting from the origin, walk(A) touches the line $L_1 : y = 3x + t$ at $(i, 3i+t)$ for the first time. If $A \in \mathscr{A}_j^-$ then walk(A) touches the line $L_2 : y = \frac{1}{3}(x - (n -$

$t)) + n$ at $(n-3j-t, n-j)$ and after passing this point this walk never touches the line again.

Let c_i be the number of walks from $(0,0)$ to $(i, 3i+t)$ which touch the line L_1 only at $(i, 3i+t)$. Then it follows that $c_i = \frac{t}{4i+t}\binom{4i+t}{i}$ (see e.g. Fact 3 in [24]). Set $\mathscr{A}_{i\bar{j}} = \mathscr{A}_i \cap \mathscr{A}_{\bar{j}}$. From now on, i and j denote some fixed constants, and we consider the situation $n \to \infty$. Then we have

$$|\mathscr{A}_{i\bar{j}}| = c_i c_j \binom{2n-2t-4(i+j)}{n-t-3i-j} \approx \frac{c_i c_j}{2^{4(i+j)}} \binom{2n-2t}{n-t}. \tag{26}$$

By Fact 2 and Fact 3 every walk corresponding to a member of $\mathscr{F}$ touches both L_1 and L_2. Thus we have $\mathscr{F} \subset \bigcup_{i,j} \mathscr{A}_{i\bar{j}}$. Set $\mathscr{F}_{i\bar{j}} = \mathscr{A}_{i\bar{j}} \cap \mathscr{F}$ and

$$\mathscr{G}_{i\bar{j}} = \{F \cap [4i+t+1, 2n-4j-t] : F \in \mathscr{F}_{i\bar{j}}\}.$$

Clearly we have $|\mathscr{F}_{i\bar{j}}| \le c_i c_j |\mathscr{G}_{i\bar{j}}|$. So we can bound $|\mathscr{F}_{i\bar{j}}|$ by bounding $|\mathscr{G}_{i\bar{j}}|$.

Claim 2. $\mathscr{G}_{0\bar{j}} \subset \binom{[t+1, 2n-t-4j]}{n-t-j}$ *is* 3*-wise* 1*-intersecting.*

Proof. Suppose on the contrary that there exist $A, B, C \in \mathscr{G}_{0\bar{j}}$ such that $A \cap B \cap C = \emptyset$. By the shiftedness we may assume that $A \cup T, B \cup T, C \cup T \in \mathscr{F}$ where $T = [t] \cup \{2n-t-4i+1 : 0 \le i < j\}$. Then using shiftedness again we may also assume that the following three subsets A', B', C' belong to $\mathscr{F}$:

$$\begin{aligned} A' &= [t] \cup A \cup \{2n-t-4i+1 : 0 \le i < j\}, \\ B' &= [t] \cup B \cup \{2n-t-4i : 0 \le i < j\}, \\ C' &= [t] \cup C \cup \{2n-t-4i-1 : 0 \le i < j\}. \end{aligned}$$

Then we have $A' \cap B' \cap C' = [t]$, which contradicts (24). □

By Claim 2 and Theorem 1 we can bound $|\mathscr{G}_{0\bar{j}}|$, and we have

$$|\mathscr{F}_{0\bar{j}}| \le c_0 c_j |\mathscr{G}_{0\bar{j}}| \le c_0 c_j \binom{2n-2t-4j-1}{n-t-j-1} \approx \frac{1}{2} |\mathscr{A}_{0\bar{j}}|. \tag{27}$$

By considering the complement we also have

$$|\mathscr{F}_{i\bar{0}}| \le \frac{1+o(1)}{2} |\mathscr{A}_{i\bar{0}}|. \tag{28}$$

Claim 3. $\mathscr{G}_{1\bar{j}} \subset \binom{[t+5, 2n-t-4j]}{n-t-j-3}$ *is* 3*-wise* 1*-intersecting.*

Proof. Suppose on the contrary that there exist $A, B, C \in \mathscr{G}_{1\bar{j}}$ such that $A \cap B \cap C = \emptyset$. By the shiftedness we may assume that the following three subsets A', B', C' belong to $\mathscr{F}$:

$$\begin{aligned} A' &= ([t+4] - \{t\}) \cup A \cup \{2n-t-4i+1 : 0 \le i < j\}, \\ B' &= ([t+4] - \{t+1\}) \cup B \cup \{2n-t-4i : 0 \le i < j\}, \\ C' &= ([t+4] - \{t+2\}) \cup C \cup \{2n-t-4i-1 : 0 \le i < j\}. \end{aligned}$$

If there exists $F \in \mathscr{F}$ such that $|F \cap [t+4]| \leq t+2$ then using the shiftedness we may assume that $F \cap [t+4] = [t+2]$. But this is impossible because $A' \cap B' \cap C' \cap F = [t-1]$, contradicting the 4-wise t-intersecting property. So we may assume that $|F \cap [t+4]| \geq t+3$ holds for all $F \in \mathscr{F}$. In other words, walk(F) passes $(0, t+4)$ or $(1, t+3)$. Since walk(F) touches the line L_2, Proposition 11 implies

$$|\mathscr{F}| \leq \alpha^t \binom{2n-t-4}{n} + (1+\varepsilon)(t+4)\alpha^t \binom{2n-t-4}{n-1} \approx (t+5)\alpha^t 2^{t-4} \binom{2n-2t}{n-t},$$

where $\alpha \approx 0.54$ is the root of the equation $X^4 - 2X + 1 = 0$. The RHS is less than $\binom{2n-2t}{n-t}$ for $t \leq 5$ and this contradicts (25). □

By Claim 3 and Theorem 1 we have

$$|\mathscr{F}_{1\bar{j}}| \leq \frac{1+o(1)}{2} |\mathscr{A}_{1\bar{j}}| \quad \text{and} \quad |\mathscr{F}_{i\bar{1}}| \leq \frac{1+o(1)}{2} |\mathscr{A}_{i\bar{1}}|. \tag{29}$$

Let I be the set of 18 pairs of indices:

$$I = \{(i,j) \in \mathbb{N}^2 : i \geq 0, j \geq 0, i+j \leq 5, \min\{i,j\} \leq 1\}.$$

By (27), (28) and (29) we have

$$\sum_{(i,j)\in I} |\mathscr{F}_{i\bar{j}}| \leq \frac{1+o(1)}{2} \sum_{(i,j)\in I} |\mathscr{A}_{i\bar{j}}|. \tag{30}$$

By Proposition 12 (setting $q = r = 4$, $s = t$ and $u = v = 0$) we have

$$\sum_{x,y} |\mathscr{A}_{x,\bar{y}}| \leq (1+o(1))\alpha^{2t} \binom{2n}{n}. \tag{31}$$

Finally, by (30), (31) and (26), we have

$$\begin{aligned}
|\mathscr{F}| &= \sum_{(i,j)\in I} |\mathscr{F}_{i\bar{j}}| + \sum_{(x,y)\notin I} |\mathscr{F}_{x,\bar{y}}| \leq \sum_{(i,j)\in I} |\mathscr{F}_{i\bar{j}}| + \sum_{(x,y)\notin I} |\mathscr{A}_{x,\bar{y}}| \\
&\leq \frac{1+o(1)}{2} \sum_{(i,j)\in I} |\mathscr{A}_{i\bar{j}}| + \left(\sum_{x,y} |\mathscr{A}_{x,\bar{y}}| - \sum_{(i,j)\in I} |\mathscr{A}_{i\bar{j}}| \right) \\
&\leq (1+o(1)) \left(\alpha^{2t} \binom{2n}{n} - \frac{1}{2} \sum_{(i,j)\in I} |\mathscr{A}_{i\bar{j}}| \right) \\
&\approx \left((2\alpha)^{2t} - \frac{1}{2} \sum_{(i,j)\in I} \frac{c_i c_j}{2^{4(i+j)}} \right) \binom{2n-2t}{n-t}.
\end{aligned}$$

Noting that $c_i = \frac{t}{4i+t}\binom{4i+t}{i}$ one can verify that the RHS is less than $0.998\binom{2n-2t}{n-t}$ for $1 \leq t \leq 4$, which contradicts (25). This completes the proof of Theorem 8. □

5. INTERSECTING SPERNER FAMILIES

Recall that an r-wise t-intersecting family $\mathscr{F} \subset 2^{[n]}$ is called non-trivial if $|\bigcap_{F\in\mathscr{F}}| < t$. Let $m^*(n,k,r,t)$ be the maximal size of k-uniform non-trivial t-intersecting families on n vertices.

Theorem 18. *Let $r \geq 4$ and t be fixed positive integers. Suppose that there exists $\gamma = \gamma(r,t) > 0$ and $\varepsilon = \varepsilon(\gamma) > 0$ such that $m^*(n,k,r,t) \leq (1-\gamma)\binom{n-t}{k-t}$ holds for $|\frac{k}{n} - \frac{1}{2}| < \varepsilon$ and $n > n_0(\varepsilon)$. Let $\mathscr{F} \subset 2^{[n]}$ be an r-wise t-intersecting Sperner family. Then we have $|\mathscr{F}| \leq \binom{n-t}{\lceil\frac{n-t}{2}\rceil}$ for $n > n_0(\varepsilon)$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[n]}{k} : [t] \subset F\}$ where $k = t + \lceil\frac{n-t}{2}\rceil$ or $k = t + \lfloor\frac{n-t}{2}\rfloor$.*

Proof of Theorem 18. Our proof is based on the idea from [4]. For a family $\mathscr{F} \subset 2^{[n]}$, set $\mathscr{F}_k = \mathscr{F} \cap \binom{[n]}{k}$. Let $\gamma > 0$ and $\varepsilon > 0$ be as in the theorem and set $K = \{k \in \mathbb{N} : (\frac{1}{2} - \varepsilon)n < k < (\frac{1}{2} + \varepsilon)n\}$. First we prove the following inequality.

Claim 4. *Let $\mathscr{F} \subset 2^{[n]}$ be a non-trivial r-wise t-intersecting Sperner family with $n > n_1(\varepsilon,\gamma)$. Then we have $\sum_{k\in K} |\mathscr{F}_k|/\binom{n-t}{k-t} < 1-\gamma$.*

Proof. First suppose that $\bigcup_{k\in K}\mathscr{F}_k$ is trivial and $[t] \subset F$ holds for all $F \in \bigcup_{k\in K}\mathscr{F}_k$. Since $\mathscr{F}$ is non-trivial, we can find $F' \in \mathscr{F}$ such that $|[t] \cap F'| < t$. Thus, for each $k \in K$, $\mathscr{F}'_k := \{F - [t] : F \in \mathscr{F}_k\}$ is $(r-1)$-wise 1-intersecting, and we have

$$|\mathscr{F}_k| = |\mathscr{F}'_k| \leq \binom{n-t-1}{k-t-1} < \frac{k}{n}\binom{n-t}{k-t} < (\frac{1}{2}+\varepsilon)\binom{n-t}{k-t},$$

which gives the desired inequality. Thus we may suppose that $\bigcup_{k\in K}\mathscr{F}_k$ is non-trivial. We prove $\sum_{k\in K}|\mathscr{F}_k|/\binom{n-t}{k-t} < 1-\gamma$ for $n > n_1$ by induction on the number of nonzero $|\mathscr{F}_k|$'s.

If this number is one then the inequality follows from the assumption of Theorem 18. If it is not the case then let i be the smallest and j the second-smallest index in K for which $|\mathscr{F}_k| \neq 0$. Set $\mathscr{F}_i^c = \{[n] - F : F \in \mathscr{F}_i\} \subset \binom{[n]}{n-i}$. Since $\mathscr{F}_i$ is r-wise t-intersecting, it follows from our assumption on $m^*(n,k,r,t)$ that $|\mathscr{F}_i| = |\mathscr{F}_i^c| \leq \binom{n-t}{i-t} = \binom{n-t}{n-i}$. Then by Proposition 13, we have

$$\frac{|\Delta_{n-j}(\mathscr{F}_i^c)|}{|\mathscr{F}_i^c|} \geq \frac{\binom{n-t}{n-j}}{\binom{n-t}{n-i}} = \frac{\binom{n-t}{j-t}}{\binom{n-t}{i-t}}. \tag{32}$$

Set $\mathscr{G}_j = \{G \in \binom{[n]}{j} : G \supset \exists F \in \mathscr{F}_i\}$. Due to (32) and the fact $\mathscr{G}_j = (\Delta_{n-j}(\mathscr{F}_i^c))^c$, we have $|\mathscr{G}_j|/\binom{n-t}{j-t} \geq |\mathscr{F}_i|/\binom{n-t}{i-t}$. Since $\mathscr{F}$ is Sperner, $\mathscr{F}_j \cap \mathscr{G}_j = \emptyset$ and $\mathscr{H} = (\mathscr{F} - \mathscr{F}_i) \cup \mathscr{G}_j$ is an r-wise t-intersecting Sperner family. Moreover, the number of nonzero $|\mathscr{H}_k|$'s is one less than that of $|\mathscr{F}_k|$'s. Therefore, by the induction hypothesis and the fact that $\mathscr{F} \triangle \mathscr{H} = \mathscr{F}_i \cup \mathscr{G}_j$, we have

$$\sum_{k\in K}\frac{|\mathscr{F}_k|}{\binom{n-t}{k-t}} \leq \sum_{k\in K}\frac{|\mathscr{H}_k|}{\binom{n-t}{k-t}} \leq 1-\gamma,$$

which completes the proof of the claim. □

We continue to prove Theorem 18. Let $\mathscr{F} \subset 2^{[n]}$ be an r-wise t-intersecting Sperner family. First suppose that $\mathscr{F}$ fixes t-element set, say $[t]$. Then $\mathscr{G} = \{F \setminus [t] : F \in \mathscr{F}\} \subset 2^{[t+1,n]}$ is a Sperner family. Thus by the Sperner Theorem [20] we have

$$|\mathscr{F}| = |\mathscr{G}| \leq \binom{n-t}{\lceil (n-t)/2 \rceil}.$$

Equality holds iff $\mathscr{G} \cong \binom{[n-t]}{\lceil (n-t)/2 \rceil}$ or $\binom{[n-t]}{\lfloor (n-t)/2 \rfloor}$.

Next suppose that $\mathscr{F}$ is non-trivial. By Claim 4, we have

$$1-\gamma > \sum_{k \in K} \frac{|\mathscr{F}_k|}{\binom{n-t}{k-t}} \geq \sum_{k \in K} \frac{|\mathscr{F}_k|}{\binom{n-t}{\lceil (n-t)/2 \rceil}}.$$

On the other hand, by the Yamamoto (or LYM) inequality [26], we have

$$1 \geq \sum_{\ell \notin K} \frac{|\mathscr{F}_\ell|}{\binom{n}{\ell}} \geq \sum_{\ell \notin K} \frac{|\mathscr{F}_\ell|}{\binom{n}{(\frac{1}{2}+\varepsilon)n}}.$$

Therefore, we have

$$|\mathscr{F}| \leq (1-\gamma)\binom{n-t}{\lceil (n-t)/2 \rceil} + \binom{n}{(\frac{1}{2}+\varepsilon)n} < \binom{n-t}{\lceil (n-t)/2 \rceil}$$

for sufficiently large n. □

Now set t_r for $4 \leq r \leq 10$ as follows.

r	4	5	6	7	8	9	10
t_r	7	18	41	89	184	377	762

By Theorem 18 and Theorem 17 we have the following result, which includes Theorem 10.

Theorem 19. *Let $4 \leq r \leq 10$, $1 \leq t \leq t_r$ and let $\mathscr{F} \subset 2^{[n]}$ be an r-wise t-intersecting Sperner family with $n > n_0$. Then we have $|\mathscr{F}| \leq \binom{n-t}{\lceil \frac{n-t}{2} \rceil}$. Equality holds iff $\mathscr{F} \cong \{F \in \binom{[n]}{k} : [t] \subset F\}$ where $k = t + \lceil \frac{n-t}{2} \rceil$ or $k = t + \lfloor \frac{n-t}{2} \rfloor$.*

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