Burning numbers via eigenpolytopes

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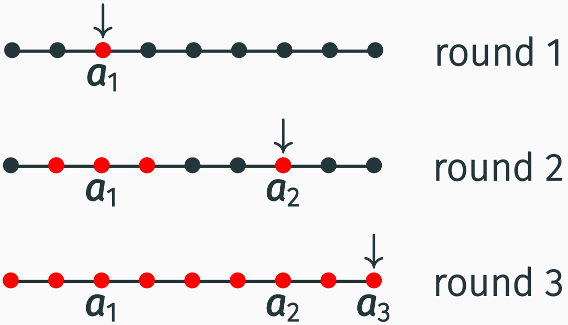
arXiv:2508.17559

1.
Burning Numbers $b(G)$.

Alon's transmitting problem

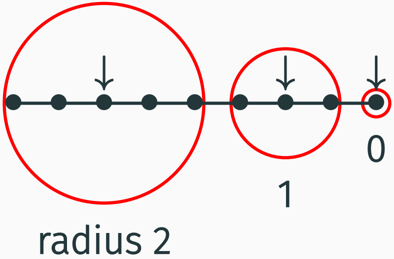
- A finite connected graph G is given.
- There is a Sender outside the graph, and Sender has a sequence of vertices $a_1, a_2, \ldots$ (called a burning sequence).
- Sender sends a message to a vertex a_i at round i.
- A vertex received the message at any round will transmit it to its neighbors at the next round.
- What is the minimum number of rounds (over all burning sequences) required for all vertices to receive the message?
- Let $b(G)$ be the minimum number (called a burning number).

What is $b(P_9)$?



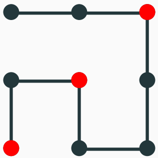
(a_1, a_2, a_3) is a burning sequence giving $b(P_9) \leq 3$.

What is $b(P_9)$?

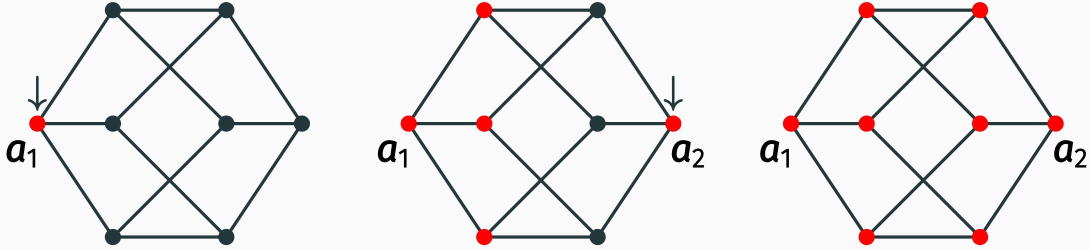


$V(P_9)$ is covered by 3 disjoint balls of radius $2, 1, 0$.

$b(P_9) = 3$. Indeed, $b(P_n) = \lceil \sqrt{n} \rceil$.



What is $b(Q_3)$?



round 1 round 2 round 3

$(a_1, a_2, *)$ is a burning sequence giving $b(Q_3) \leq 3$.

Some facts about burning numbers

- $b(G)$ was first introduced by Noga Alon (1992). Recently the concept has been rediscovered and called "burning number."
- Graph burning process is a model for the spread of influence.
- $b(K_n) = 2,\ \ b(P_n) = \lceil \sqrt{n} \rceil,\ \ b(G) \leq \mathrm{diam}(G) + 1.$
- Burning number conjecture (Bonato et al. 2016)
  For every connected n-vertex graph G, $b(G) \leq \lceil \sqrt{n} \rceil$.
- Graph burning problem is NP-complete.
  (Instance) n-vertex graph G and $k \geq 2$. (Question) Is $b(G) \leq k$?
- Alon obtained the burning number of the hypercube graphs.

Let Q_n denote the n-dim hypercube, that is,

- $V(Q_n) = \{0,1\}^n = \{(v_1, \dots, v_n) : v_i \in \{0,1\}\}$, and
- two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent if $\#\{i : u_i \neq v_i\} = 1$.

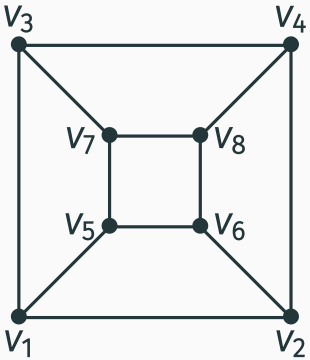
Theorem (Alon 1992)

$b(Q_n) = \lceil n/2 \rceil + 1$.

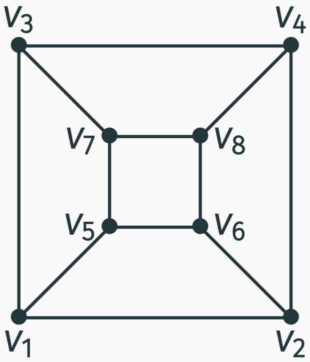
- $b(Q_n) \le \lceil \frac{n}{2} \rceil + 1$ is easy. Let $\boldsymbol{a}_1 = \boldsymbol{0}$ and $\boldsymbol{a}_2 = \boldsymbol{1}$.
- To get the lower bound, Alon used the fact that Q_n is realized as a polytope in $\mathbb{R}^n$.

2.
Eigenpolytopes

A toy example for an eigenpolytope



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$



$$A = \left[\begin{array}{cccc|cccc} & 1 & 1 & & 1 & & & \\ 1 & & & 1 & & 1 & & \\ 1 & & & 1 & & & 1 & \\ & 1 & 1 & & & & & 1 \\ \hline 1 & & & & & 1 & 1 & \\ & 1 & & & 1 & & & 1 \\ & & 1 & & 1 & & & 1 \\ & & & 1 & & 1 & 1 & \end{array}\right]$$

Eigenvalues are $3, 1, 1, 1, -1, -1, -1, -3$.

Let $\theta = 1$. (the 2nd largest eigenvalue)

$\mathbf{dim}(\theta\text{-eigenspace}) = 3$.

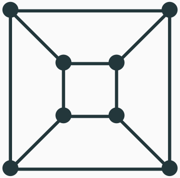
column vectors are ONB (orthonormal basis) of θ-eigenspace

$$B = \frac{1}{\sqrt{8}} \left[\begin{array}{r|r|r} -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

row vectors determine 8 points in $\mathbb{R}^3$ whose convex hull is the cube

$$B = \frac{1}{\sqrt{8}} \left[\begin{array}{rrr} -1 & -1 & -1 \\ \hline -1 & 1 & -1 \\ \hline -1 & -1 & 1 \\ \hline -1 & 1 & 1 \\ \hline 1 & -1 & -1 \\ \hline 1 & 1 & -1 \\ \hline 1 & -1 & 1 \\ \hline 1 & 1 & 1 \end{array}\right]$$

a graph G

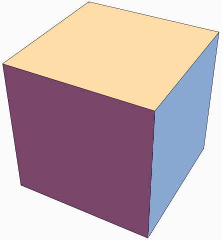


$\Downarrow$

θ-eigenspace (θ is the 2nd largest eigenvalue)

$\Downarrow$

a polytope $\mathcal{P}$



Formal definition of an eigenpolytope

- Let G be an n-vertex graph.
- Let θ be the 2nd largest eigenvalue, and m be its multiplicity.
- Let B be an $n \times m$ matrix whose columns are ONB of the θ-eigenspace.
- Let $\mathcal{P} \subset \mathbb{R}^m$ be the convex hull of the n rows of B.
- $\mathcal{P}$ depends on the choice of ONB, but the inner product of any two vertices of $\mathcal{P}$ is independent of the choice.
- We call $\mathcal{P}$ the eigenpolytope of G.

Theorem (Godsil 1998, Eigenpolytopes of distance regular graphs)

Let G be distance regular and let $\mathcal{P}$ be the eigenpolytope of G. Then G is the 1-skeleton of $\mathcal{P}$ if and only if it is one of the following:

(a) a Hamming graph $H(n, q)$,

(b) a Johnson graph $J(n, k)$,

(c) a halved n-cube $\frac{1}{2}H(n, 2)$,

(d) the Schläfli graph, (e) the Gosset graph, (f) the icosahedron, (g) the dodecahedron, (h) the complement of r copies of K_2, or (i) a cycle C_n.

cf. The n-dim hypercube Q_n is $H(n, 2)$.

3.
Results and proof ideas

Our results (Tanaka-T 2025+)

(a) For Hamming graphs, we have

$$\left\lfloor \left(1-\frac{1}{q}\right) n \right\rfloor + 1 \leq b(H(n,q)) \leq \left\lfloor \left(1-\frac{1}{q}\right) n + \frac{q+1}{2} \right\rfloor .$$

(b) For Johnson graphs, we have

$$b(J(n,k)) = k+1 \text{ for } n > k^2,$$
$$b(J(2k,k)) = \lceil k/2 \rceil + 1.$$

(c) For halved n-cubes, we have

$$b(\tfrac{1}{2}H(n,2)) = \lceil n/4 \rceil + 1.$$

Outline of proof

- Upper bounds:
  - We construct a burning sequence. This part is easy.
- Lower bounds:
  - We want to show that $b(G) > b$.
  - To this end, for any given burning sequence $a_1, a_2, \ldots, a_b$, we will find a vertex x which remains unburned at round b.
  - We give an algorithm finding such vertex by solving a system of linear equations repeatedly on the faces of the eigenpolytope.
  - I'll explain the algorithm using the case $b(Q_4)$ as a toy example.

How to prove $b(Q_4) > 2$?

For any given burning sequence (a_1, a_2), we need to find a vertex x which remains unburned at round 2.

- Let $a_1, a_2 \in V(Q_4)$ be given.
- Want: $\exists x \in V(Q_4)$ such that $d(a_1, x) > 1$, $d(a_2, x) > 0$.

Let $\mathcal{P}_n$ be the eigenpolytope of Q_n.
Put $\mathcal{P}_4 \subset \mathbb{R}^4$ so that all the vertices are on the unit sphere centered at $\mathbf{0}$.

- Let $\boldsymbol{a}_1, \boldsymbol{a}_2 \in V(\mathcal{P}_4)$ be given.
- Want: $\exists \boldsymbol{x} \in V(\mathcal{P}_4)$ such that $\boldsymbol{a}_1 \cdot \boldsymbol{x} \leq 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 1/2$.

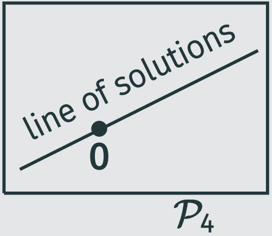
The algorithm: $b(Q_4) > 2$?

- Put $\mathcal{P}_4 \subset \mathbb{R}^4$ so that all the vertices are on the unit sphere centered at $\mathbf{0}$.
- Let $\boldsymbol{a}_1, \boldsymbol{a}_2 \in V(\mathcal{P}_4)$ be given.
- We show $\exists \boldsymbol{x} \in V(\mathcal{P}_4)$ such that $\boldsymbol{a}_1 \cdot \boldsymbol{x} \leq 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 1/2$.
- Choose $\boldsymbol{a}_3 \in V(\mathcal{P}_4)$ arbitrarily.
- For $i = 3, 2, 1, 0$, we will find an i-dim face $F_i \subset \mathcal{P}_4$ and a point $\boldsymbol{x}_i \in F_i$ such that $\boldsymbol{a}_i \cdot \boldsymbol{x}_i = 0$.
- Then $\boldsymbol{x} := \boldsymbol{x}_0$ satisfies the required conditions.

- Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in V(\mathcal{P}_4)$ be given. (coefficient vectors)
- For $i = 3, 2, 1, 0$, we will find an i-dim face $F_i \subset \mathcal{P}_4$ and a point $\boldsymbol{x}_i \in F_i$ such that $\boldsymbol{a}_i \cdot \boldsymbol{x}_i = 0$.

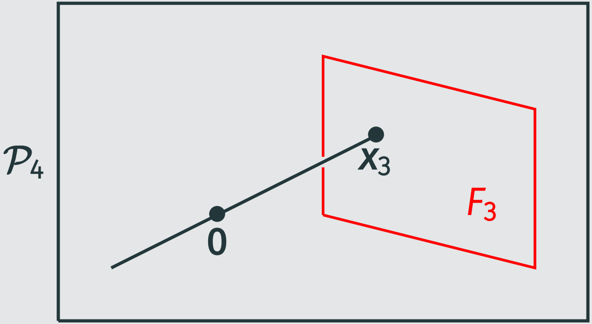
Step 1: ($i = 3$)

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{z} = 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{z} = 0$, $\boldsymbol{a}_3 \cdot \boldsymbol{z} = 0$ in $\mathbb{R}^4$.
- $\boldsymbol{z} = \boldsymbol{0} = (0, 0, 0, 0)$ is one of the solutions.
- 3 equations, 4 variables ($\boldsymbol{z} \in \mathbb{R}^4$).
- There is a line of solutions passing through $\boldsymbol{0}$.



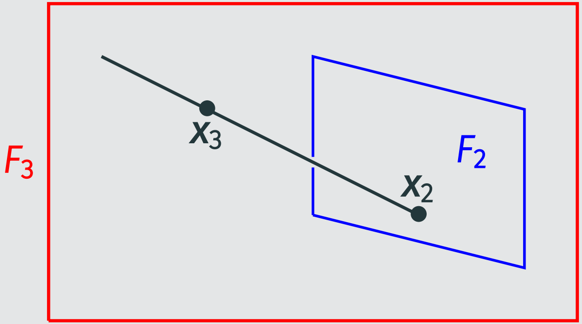
Step 1: ($i = 3$)

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{z} = 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{z} = 0$, $\boldsymbol{a}_3 \cdot \boldsymbol{z} = 0$ in $\mathbb{R}^4$.
- move along the line until we hit a facet $F_3 \cong \mathcal{P}_3$.
- Let $\boldsymbol{x}_3 \in F_3$ be a solution of the intersection. (so $\boldsymbol{a}_3 \cdot \boldsymbol{x}_3 = 0$)



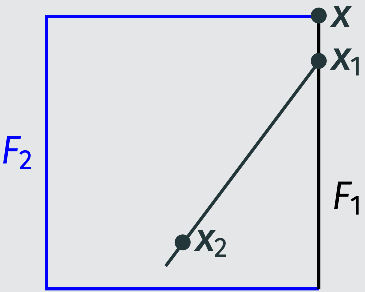
Step 2: ($i = 2$)

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{z} = 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{z} = 0$ in F_3.
- move along the line until we hit a face $F_2 \cong \mathcal{P}_2$.
- Let $\boldsymbol{x}_2 \in F_2$ be the intersection. (so $\boldsymbol{a}_2 \cdot \boldsymbol{x}_2 = 0$)



Step 3 and 4: ($i = 1$ and 0)

- Solve $\boldsymbol{a}_1 \cdot \boldsymbol{z} = 0$ in F_2.
- move along the line until we hit an edge $F_1 \cong \mathcal{P}_1$.
- Let $\boldsymbol{x}_1 \in F_1$ be the intersection. (so $\boldsymbol{a}_1 \cdot \boldsymbol{x}_1 = 0$)
- Let $\boldsymbol{x}$ be one of the endpoints of F_1.
- Then $\boldsymbol{x} \in F_1 \subset F_2$.
- The $\boldsymbol{x}$ is the desired vertex, that is, $\boldsymbol{a}_1 \cdot \boldsymbol{x} \leq 0$, $\boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 1/2$.



Why $\boldsymbol{a}_i \cdot \boldsymbol{x}$ is small in general ?

- We have $\boldsymbol{x}_i, \boldsymbol{x} \in F_i \cong \mathcal{P}_i$.
- Then, $|\boldsymbol{x}_i - \boldsymbol{x}| \le \mathrm{diam}(\mathcal{P}_i)$.
- So, $\boldsymbol{x}_i$ and $\boldsymbol{x}$ are close (depending on diam), and
- $\boldsymbol{a}_i \cdot \boldsymbol{x}_i$ and $\boldsymbol{a}_i \cdot \boldsymbol{x}$ are close, say,

  $$\boldsymbol{a}_i \cdot \boldsymbol{x} \le \boldsymbol{a}_i \cdot \boldsymbol{x}_i + \epsilon$$

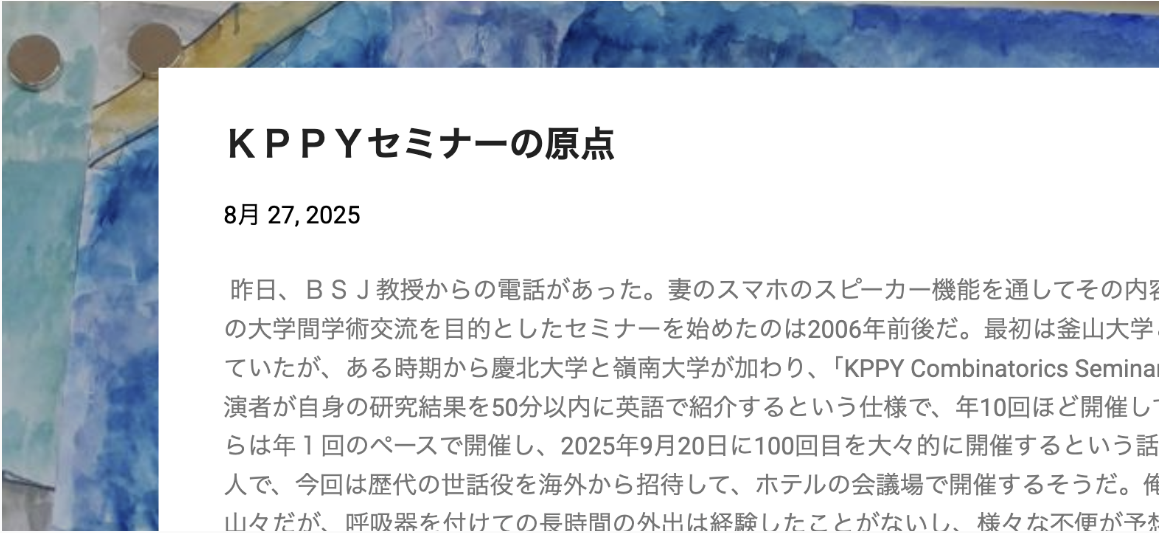
  for some ϵ (depending on $\mathrm{diam}(\mathcal{P}_i)$).
- Since $\boldsymbol{a}_i \cdot \boldsymbol{x}_i = 0$, we have $\boldsymbol{a}_i \cdot \boldsymbol{x} \le \epsilon$. $\square$

Background and References

- This search algorithm was introduced by Beck and Fiala. (Integer making theorem, 1981)
- Beck and Spencer used it to solve a Berlekamp's puzzle. (Balancing matrices with line shifts, 1983).
- Alon used a lemma by Beck and Spencer to get $b(Q_n)$. (Transmitting in the n-dimensional cube, 1992)
- To bound $\boldsymbol{a}_i \cdot \boldsymbol{x}$, Alon used a fixed coordinate system. This works well for Q_n.
- For $H(n, q)$ and $J(n, k)$, we used cosine of the angle between $\boldsymbol{a}_i$ and $\boldsymbol{x}$ based on Godsil's observation.

Problems

- What is the exact burning number of $H(n,q)$ and $J(n,k)$?
- Is $b(H(3s,3)) = 2s+2$? We know $2s+1 \leq b \leq 2s+2$.
- Other graphs ?
- Alternative proofs ?



ＫＰＰＹセミナーの原点

8月 27, 2025

昨日、ＢＳＪ教授からの電話があった。妻のスマホのスピーカー機能を通してその内
の大学間学術交流を目的としたセミナーを始めたのは2006年前後だ。最初は釜山大学
ていたが、ある時期から慶北大学と嶺南大学が加わり、「KPPY Combinatorics Semina
演者が自身の研究結果を50分以内に英語で紹介するという仕様で、年10回ほど開催し
らは年１回のペースで開催し、2025年9月20日に100回目を大々的に開催するという
人で、今回は歴代の世話役を海外から招待して、ホテルの会議場で開催するそうだ。
山々だが、呼吸器を付けての長時間の外出は経験したことがないし、様々な不便が予

hirasakajuku.blogspot.com/2025/08/blog-post_27.html

KPPY organizers

Mitsugu Hirasaka, Jack Koolen, Tommy Jensen, Mark Siggers, Sejeong Bang, Alex Gavrilyuk, Jongyook Park, Jeong Rye Park, Semin Oh, Jihye Park