CROSS t-INTERSECTING INTEGER SEQUENCES FROM WEIGHTED ERDŐS–KO–RADO

NORIHIDE TOKUSHIGE

Abstract. Let m, n and t be positive integers. Consider $[m]^n$ as the set of sequences of length n on an m-letter alphabet. We say that two subsets $A \subset [m]^n$ and $B \subset [m]^n$ cross t-intersect if any two sequences $a \in A$ and $b \in B$ match in at least t positions. In this case it is shown that if $m > (1 - \frac{1}{\sqrt[t]{2}})^{-1}$ then $|A||B| \leq (m^{n-t})^2$. We derive this result from a weighted version of the Erdős–Ko–Rado theorem concerning cross t-intersecting families of subsets, and we also include the corresponding stability statement. One of our main tools is the eigenvalue method for intersection matrices due to Friedgut [10].

(2010 AMS subject classification codes 05D05, 05C50.)

1. Introduction

Hoffman observed that one can get an upper bound of the independence number of a given regular graph by using the eigenvalues of the adjacency matrix. This eigenvalue method has been extended in various ways with many applications. For example, this approach gave the exact bound for parameters in the Erdős–Ko–Rado theorem [5]. Namely, Wilson [14] obtained the maximum size of n-vertex k-uniform t-intersecting families, which is $\binom{n-t}{k-t}$ if $n \geq (t+1)(k-t+1)$, and Frankl and Wilson [9] obtained the corresponding vector space analogue. Recently, Ellis, Friedgut and Pilpel [4] succeeded to determine the maximum size of n-letter t-intersecting permutations, which is $(n-t)!$ if $n \gg t$. As is pointed out in [4], it is one of the merits of the eigenvalue method that one can modify a proof for t-intersecting result only slightly to get the corresponding stronger "cross t-intersecting" result. In fact, Ellis, Friedgut and Pilpel obtained $((n-t)!)^2$ bound for the product of the sizes of two families of cross t-intersecting permutations. The same method can apply to get $\binom{n-t}{k-t}^2$ bound for n-vertex k-uniform cross t-intersecting families, and its vector space analogue, see [16]. Along this line, in this article, we will determine the maximum product of the sizes of two cross t-intersecting sets of integer sequences. Gromov [11] claims that such inequalities can be equivalently reformulated in terms of monomial subsets in the N-torus, and for example he obtained a homological separation inequality for pairs of disjoint subsets in N-torus from a cross intersecting Erdős–Ko–Rado inequality obtained in [13].

Date: November 24, 2013.

2010 *Mathematics Subject Classification.* Primary: 05D05, Secondary: 05C50.

Key words and phrases. cross intersecting family; eigenvalue method.

The author was supported by JSPS KAKENHI 20340022.

Let n, t be positive integers with $n \geq t$, and let $p, q \in (0,1)$ be reals satisfying $p + q = 1$. Let $[n] := \{1, 2, \ldots, n\}$ and let $\mu_p : 2^{[n]} \to [0,1]$ be the product measure defined by

$$\mu_p(x) := p^{|x|} q^{n-|x|}$$

for a subset $x \subset [n]$. For a family of subsets $\mathcal{F} \subset 2^{[n]}$ define the measure of $\mathcal{F}$ by

$$\mu_p(\mathcal{F}) := \sum_{x \in \mathcal{F}} \mu_p(x).$$

We say that $\mathcal{F} \subset 2^{[n]}$ is t-intersecting if $|x \cap x'| \geq t$ holds for all $x, x' \in \mathcal{F}$. What is the maximum measure for t-intersecting families? To answer this question, let us define a t-intersecting family $\mathcal{A}_r(n,t) \subset 2^{[n]}$ by

$$\mathcal{A}_r(n,t) := \{A \subset [n] : |A \cap [t+2r]| \geq t + r\}.$$

The following result was first proved by Ahlswede and Khachatrian [1], see also Bey and Engel [2], Dinur and Safra [3], and Tokushige [15].

Theorem 1 ([1, 2, 3, 15]). *Let $n \geq t \geq 1$ be integers and let $p \in (0,1)$. If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then $\mu_p(\mathcal{F}) \leq \max_r \mu_p(\mathcal{A}_r(n,t))$. If equality holds then $\mathcal{F}$ is isomorphic to one of $\mathcal{A}_r(n,t)$'s.*

Consider the case $p \in (0, \frac{1}{t+1})$. Then $\max_r \mu_p(\mathcal{A}_r(n,t)) = \mu_p(\mathcal{A}_0(n,t)) = p^t$. Friedgut [10] gave a proof of this case of Theorem 1 using the eigenvalue method. Furthermore he also showed the stability of the extremal structure described as follows.

Theorem 2 ([10]). *Let $n \geq t \geq 1$ be integers and let $p \in (0, \frac{1}{t+1})$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is t-intersecting. Then the following holds.*

(i) *We have $\mu_p(\mathcal{F}) \leq p^t$ with equality holding iff $\mathcal{F}$ is isomorphic to $\mathcal{A}_0(n,t)$.*
(ii) *There is a constant $c = c(t,p)$ such that if $\mu_p(\mathcal{F}) > (1-\epsilon)p^t$ then $\mu_p(\mathcal{F} \triangle \mathcal{G}) < c\epsilon$ for some $\mathcal{G} \cong \mathcal{A}_0(n,t)$.*

We first extend (i) of the above result to cross t-intersecting families. We say that two families of subsets $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ are cross t-intersecting if $|x \cap y| \geq t$ holds for all $x \in \mathcal{F}_1, y \in \mathcal{F}_2$. Let $\binom{[n]}{k}$ denote the set of all k-subsets of $[n]$.

Theorem 3. *Let $n \geq t \geq 1$ be integers, let $p \in (0, 1 - \frac{1}{\sqrt[t]{2}})$, and let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting, then $\mu_p(\mathcal{F}_1)\mu_p(\mathcal{F}_2) \leq p^{2t}$ with equality holding iff $\mathcal{F}_1 = \mathcal{F}_2 \cong \mathcal{A}_0(n,t)$, that is, $\mathcal{F}_1 = \mathcal{F}_2 = \{F \subset [n] : T \subset F\}$ for some $T \in \binom{[n]}{t}$.*

For comparison we mention the corresponding k-uniform version from [16].

Theorem 4 ([16]). *Let $n \geq k \geq t \geq 1$ be integers, let $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}}$, and let $\mathcal{F}_1, \mathcal{F}_2 \subset \binom{[n]}{k}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting, then $|\mathcal{F}_1||\mathcal{F}_2| \leq \binom{n-t}{k-t}^2$ with equality holding iff $\mathcal{F}_1 = \mathcal{F}_2 \cong \mathcal{A}_0(n,t) \cap \binom{[n]}{k}$, that is, $\mathcal{F}_1 = \mathcal{F}_2 = \{F \in \binom{[n]}{k} : T \subset F\}$ for some $T \in \binom{[n]}{t}$.*

Next we extend (ii) of Theorem 2 to cross t-intersecting families.

Theorem 5. *Let $t \geq 1$ be an integer and let $p \in (0, 1 - \frac{1}{\sqrt[t]{2}})$. Then there is a constant $c = c(t,p)$ which satisfies the following. Let $n \geq t$ be an integer and let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting families with $\sqrt{\mu_p(\mathcal{F}_1)\mu_p(\mathcal{F}_2)} > (1-\epsilon)p^t$, then there is a family $\mathcal{G} \cong \mathcal{A}_0(n,t)$ such that $\mu_p(\mathcal{F}_1 \triangle \mathcal{G}) < c\sqrt{\epsilon}$ and $\mu_p(\mathcal{F}_2 \triangle \mathcal{G}) < c\sqrt{\epsilon}$.*

Finally we consider a t-intersecting set of integer sequences. Let n, m, t be positive integers with $m \geq 2$ and $n \geq t$. Then $H \subset [m]^n$ is a set of integer sequences $(a_1, \ldots, a_n)$, $1 \leq a_i \leq m$. We say that H is t-intersecting if any two sequences intersect in at least t positions, more precisely, $\#\{i : a_i = b_i\} \geq t$ holds for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in H$. Let $f_1(n,m,t)$ be the maximum size of $H \subset [m]^n$ which is t-intersecting. The following result was proved by Alhswede and Khachatrian [1] and Frankl and Tokushige [8] independently, see also Bey and Engel [2].

Theorem 6 ([1, 8, 2])**.** *Let $r = \lfloor \frac{t-1}{m-2} \rfloor$. If $n \geq t+2r$ then $f_1(n,m,t) = m^n \mu_{\frac{1}{m}}(\mathcal{A}_r(n,t))$.*

We notice that

$$m^n \mu_{\frac{1}{m}}(\mathcal{A}_0(n,t)) = \sum_{x \in \mathcal{A}_0} (m-1)^{|x|} = \sum_{k=0}^{n-t} \binom{n-t}{k} 1^k (m-1)^{n-t-k} = m^{n-t}.$$

Frankl and Füredi [7] had settled the following case, which was a starting point of the research resulting Theorem 6.

Corollary 1 ([7])**.** *If $n \geq t \geq 1$ and $m \geq t+1$, then $f_1(n,m,t) = m^{n-t}$.*

We extend this result to cross t-intersecting sets of integer sequences. For $H_1, H_2 \subset [m]^n$ we say that they are cross t-intersecting if $\#\{i : a_i = b_i\} \geq t$ holds for all $(a_1, \ldots, a_n) \in H_1$ and $(b_1, \ldots, b_n) \in H_2$. Let $f_2(n,m,t)$ be the maximum of $|H_1||H_2|$, where $H_1, H_2 \subset [m]^n$ run over all cross t-intersecting sets of integer sequences. The next result shows that this function f_2 is closely related to the measure μ_p ($p = 1/m$) as was the case with f_1.

Theorem 7. *We have $f_2(n,m,t) = \max\{m^{2n} \mu_{\frac{1}{m}}(\mathcal{F}_1)\mu_{\frac{1}{m}}(\mathcal{F}_2)\}$, where $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ run over all cross t-intersecting families.*

Consequently we obtain the following result from Theorem 3.

Theorem 8. *If $n \geq t \geq 1$ and $m > (1 - \frac{1}{\sqrt[t]{2}})^{-1}$, then $f_2(n,m,t) = (m^{n-t})^2$.*

It seems very likely that the best possible upper bound for p in Theorems 3 and 5 is $\frac{1}{t+1}$ instead of $1 - \frac{1}{\sqrt[t]{2}}$. More precisely, we conjecture the following.

Conjecture 1. *Let $n \geq t \geq 1$ be integers, let $p_1, p_2 \in (0, \frac{1}{t+1})$, and let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting, then $\mu_{p_1}(\mathcal{F}_1)\mu_{p_2}(\mathcal{F}_2) \leq (p_1p_2)^t$ with equality holding iff $\mathcal{F}_1 = \mathcal{F}_2 \cong \mathcal{A}_0(n,t)$, that is, $\mathcal{F}_1 = \mathcal{F}_2 = \{F \subset [n] : T \subset F\}$ for some $T \in \binom{[n]}{t}$.*

Conjecture 2. *Let $t \geq 1$ be an integer and let $p_1, p_2 \in (0, \frac{1}{t+1})$. Then there is a constant $c = c(t, p_1, p_2)$ which satisfies the following. Let $n \geq t$ be an integer and let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting families with $\mu_{p_1}(\mathcal{F}_1)\mu_{p_2}(\mathcal{F}_2) > (1-\epsilon)^2(p_1p_2)^t$, then there is a family $\mathcal{G} \cong \mathcal{A}_0(n,t)$ such that $\mu_{p_1}(\mathcal{F}_1 \triangle \mathcal{G}) < c\sqrt{\epsilon}$ and $\mu_{p_2}(\mathcal{F}_2 \triangle \mathcal{G}) < c\sqrt{\epsilon}$.*

Conjecture 3. *If $n \ge t \ge 1$ and $m \ge t+1$, then $f_2(n,m,t) = (m^{n-t})^2$.*

Very recently, Frankl, Lee, Siggers, and Tokushige [6] proved Conjectures 1 and 3 for the case when $p_1 = p_2$ and $t \ge 14$. Their approach is completely different from ours. See also [16] for some related problems concerning "k-uniform" cross t-intersecting families.

2. Inequality and uniqueness: Proof of Theorem 3

We follow the proof in [10] and [4]. In fact, the key observation, Lemma 1, is essentially due to Friedgut [10], and we include a (slightly more direct) proof for convenience.

Let $G = (V, E)$ be a graph with $V = 2^{[n]}$ and $E = \{\{x, y\} \in \binom{V}{2} : |x \cap y| < t\}$. Recall that μ_p is the product measure on V. In this graph, we have $\mathcal{F}_1, \mathcal{F}_2 \subset V$ and there are no edges between them, namely, $\{\{x, y\} : x \in \mathcal{F}_1,\, y \in \mathcal{F}_2\} \cap E = \emptyset$, because $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting. For $i = 1, 2$, let f_i be the characteristic function of $\mathcal{F}_i$, that is, $f_i(x) = 1$ if $x \in \mathcal{F}_i$ and $f_i(x) = 0$ if $x \in V \setminus \mathcal{F}_i$. Let

$$\alpha_i := \mu_p(\mathcal{F}_i) = \sum_{x \in \mathcal{F}_i} \mu_p(x)$$

be the measure of $\mathcal{F}_i$. We shall show that $\alpha_1\alpha_2 \le p^{2t}$.

We will define a pseudo adjacency matrix of the graph G. First let

$$A^{(1)} = \begin{pmatrix} 1 + c - cX & -c + cX \\ 1 - X & X \end{pmatrix},$$

where $c := -p/q$. Both rows and columns of $A^{(1)}$ are indexed by subsets of [1], namely, $\emptyset$ and $\{1\}$ itself. The eigenvalues of $A^{(1)}$ are 1 and $c(1 - \frac{1}{p}X)$, and the corresponding eigenvectors are $(1,1)^T$ and $(\sqrt{-c}, -\sqrt{-1/c})^T$, respectively. Next let

$$A^{(n)} = A^{(1)} \otimes \cdots \otimes A^{(1)}$$

be a $2^n \times 2^n$ matrix obtained by taking n-fold tensor of $A^{(1)}$ over the ring $\mathbb{R}[X]/(X^t)$. Then $A^{(n)} = (a_{xy})$ is a pseudo adjacency matrix of G, that is, $a_{xy} = 0$ whenever $\{x, y\} \notin E$. (One can verify this by induction on n, see [10] for details.) Then the set of eigenvalues $\{\lambda_x\}_{x \in V}$ of the matrix $A^{(n)}$ is given by

$$\lambda_x := c^{|x|}(1 - \tfrac{1}{p}X)^{|x|} = c^{|x|} \sum_{m=0}^{t-1} \lambda_x^{(m)} X^m, \tag{1}$$

where $\lambda_x^{(m)} := \binom{|x|}{m}(-\frac{1}{p})^m$. (We define $\binom{a}{b} = 0$ if $a < b$, so $\lambda_x^{(m)} = 0$ if $|x| < m$. Thus the sum in the RHS of (1) is actually taken over $m = 0, 1, \ldots, \min\{|x|, t-1\}$.) For $1 \le i \le n$ define the function $\chi_{\{i\}} : V \to \mathbb{R}$ by

$$\chi_{\{i\}}(x) := \begin{cases} \sqrt{-c} & \text{if } i \notin x \\ -\sqrt{-1/c} & \text{if } i \in x. \end{cases}$$

Then the eigenvector χ_x corresponding to λ_x is given by $\chi_x := \prod_{i \in x} \chi_{\{i\}}$. For $x = \emptyset$ we let $\chi_\emptyset := \mathbf{1}_V$ (all one vector of length 2^n).

We introduce an inner product on $\mathcal{H}_n := \{f : V \to \mathbb{R}\}$, the set of real-valued functions on V, by

$$\langle f, g\rangle_p := \sum_{x\in V} f(x)g(x)\mu_p(x).$$

Then the set of eigenvectors $\{\chi_x\}_{x\in V}$ of $A^{(n)}$ forms an orthonormal system of the inner product space $\mathcal{H}_n$. Thus we can expand the characteristic function f_i as $f_i = \sum_{x\in V} \hat{f}_i(x)\chi_x$ where $\hat{f}_i(x) = \langle f_i, \chi_x\rangle_p$.

Claim 1. *The Fourier coefficients* $\hat{f}_i(x)$*'s satisfy the following properties.*

(C1) $\alpha_i = \hat{f}_i(\emptyset)$.
(C2) $\alpha_i = \sum_{x\in V} \hat{f}_i(x)^2$.
(C3) $A^{(n)} f_i = \sum_{x\in V} \hat{f}_i(x)\lambda_x\chi_x$.
(C4) $\sum_{x\in V} \hat{f}_1(x)\hat{f}_2(x)\lambda_x = 0$.
(C5) $\sum_{x\in V} c^{|x|}\binom{|x|}{m}\hat{f}_1(x)\hat{f}_2(x) = 0$ *for* $0 \le m \le t-1$.

Proof. We write $\langle f, g\rangle$ and μ instead of $\langle f, g\rangle_p$ and μ_p for simplicity. Recall that $\chi_\emptyset = \mathbf{1}_V$. By computing $|f_i|_1$ and $|f_i|_2^2$ we have (C1) and (C2), respectively:

$$\alpha_i = \sum_{x\in\mathcal{F}_i} \mu(x) = \sum_{x\in V} f_i(x)\chi_\emptyset(x)\mu(x) = \langle f_i, \chi_\emptyset\rangle = \hat{f}_i(\emptyset),$$

$$\alpha_i = \sum_{x\in\mathcal{F}_i} \mu(x) = \sum_{x\in V} f_i(x)^2\mu(x) = \langle f_i, f_i\rangle = \langle\sum_{x\in V} \hat{f}_i(x)\chi_x, \sum_{x\in V} \hat{f}_i(x)\chi_x\rangle = \sum_{x\in V} \hat{f}_i(x)^2.$$

(C3) follows from $A^{(n)}\chi_x = \lambda_x\chi_x$ and $f_i = \sum_{x\in V} \hat{f}_i(x)\chi_x$.

To show (C4) we compute $\langle f_1, A^{(n)} f_2\rangle$ in two ways. On one hand, it follows from (C3) that

$$\langle f_1, A^{(n)} f_2\rangle = \langle\sum_{x\in V} \hat{f}_1(x)\chi_x, \sum_{x\in V} \hat{f}_2(x)\lambda_x\chi_x\rangle = \sum_{x\in V} \hat{f}_1(x)\hat{f}_2(x)\lambda_x,$$

which is the LHS of (C4). On the other hand, we have

$$\begin{aligned}\langle f_1, A^{(n)} f_2\rangle &= \sum_{x\in V} f_1(x)\big((A^{(n)} f_2)(x)\big)\mu(x) = \sum_{x\in V} f_1(x)\big(\sum_{y\in V} a_{x,y} f_2(y)\big)\mu(x)\\ &= \sum_{x\in V}\sum_{y\in V} a_{x,y} f_1(x) f_2(y)\mu(x) = 0.\end{aligned}$$

This is because if $\{x, y\} \notin E$ then $a_{x,y} = 0$ by the fact that $A^{(n)}$ is a pseudo adjacency matrix, and if $\{x, y\} \in E$ then $f_1(x)f_2(y) = 0$ by the cross t-intersecting property of $\mathcal{F}_1$ and $\mathcal{F}_2$.

By (1) and (C4) we have

$$0 = \sum_{x\in V} \hat{f}_1(x)\hat{f}_2(x)c^{|x|}\sum_{m=0}^{t-1} \lambda_x^{(m)} X^m = \sum_{m=0}^{t-1} \big(-\frac{1}{p}\big)^m X^m \sum_{x\in V} c^{|x|}\binom{|x|}{m}\hat{f}_1(x)\hat{f}_2(x).$$

So, for each m, the coefficient of X^m in the RHS vanishes and (C5) follows. □

For any $Q(Y) \in \mathbb{R}[Y]/(Y^t)$ we can write $Q(Y) = \sum_{m=0}^{t-1} \beta_m \binom{Y}{m}$ for some $\beta_0, \dots, \beta_{t-1} \in \mathbb{R}$. Then, from (C5), we have

$$0 = \sum_{m=0}^{t-1} \beta_m \Big(\sum_{x \in V} c^{|x|} \binom{|x|}{m} \hat{f}_1(x) \hat{f}_2(x) \Big) = \sum_{x \in V} c^{|x|} Q(|x|) \, \hat{f}_1(x) \hat{f}_2(x). \tag{2}$$

For $0 \le i \le n$, let

$$\theta_i := c^i Q(i),$$

which will play the role of an "eigenvalue." By (2) with (C1) we have

$$\alpha_1 \alpha_2 \theta_0 = - \sum_{x \neq \emptyset} \theta_{|x|} \, \hat{f}_1(x) \hat{f}_2(x). \tag{3}$$

Now let $Q(Y) \in \mathbb{R}[Y]/(Y^t)$ be the unique polynomial of degree $t-1$ such that $Q(i) = -c^{-i}$ for $i = 1, \dots, t$, that is,

$$\theta_1 = \theta_2 = \cdots = \theta_t = -1. \tag{4}$$

Lemma 1. *The polynomial $Q(Y)$ satisfies the following properties about $\theta_i = c^i Q(i)$.*

(Q1) $\theta_0 = p^{-t} - 1$.

(Q2) *If $q^{-t} < 2$, then $-1 \le \theta_i < 1$ for all $1 \le i \le n$.*

Proof. By the extrapolation form, we have

$$Q(Y) = - \sum_{i=1}^{t} \prod_{j \in [t] \setminus \{i\}} \frac{Y - j}{i - j} c^{-i}. \tag{5}$$

Let $d = -1/c = q/p > 1$. For (Q1), by setting $Y = 0$ in (5), we have

$$\begin{aligned} \theta_0 = c^0 Q(0) &= - \sum_{i=1}^{t} \frac{(-1)^t t!}{(i-1)!(-1)^{t-i}(t-i)!(-i)} c^{-i} = \sum_{i=1}^{t} \binom{t}{i} d^i \\ &= \sum_{i=0}^{t} \binom{t}{i} d^i 1^{t-i} - 1 = (d+1)^t - 1 = p^{-t} - 1. \end{aligned}$$

Next we show (Q2). We have $\theta_i = -1$ for $1 \le i \le t$ by (4). Now we will show that

$$1 > |\theta_{t+1}| > |\theta_{t+2}| > \cdots > |\theta_n|. \tag{6}$$

For $Y \ge t+1$ it follows from (5) that

$$-Q(Y) = \sum_{i=1}^{t} \frac{(Y-1)\cdots(Y-t)\, c^{-i}}{(i-1)!(-1)^{t-i}(t-i)!(Y-i)} = (-1)^t \sum_{i=1}^{t} \frac{(Y-1)\cdots(Y-t)\, d^i}{(i-1)!(t-i)!(Y-i)}. \tag{7}$$

Claim 2. *If $q^{-t} < 2$, then $0 < \theta_{t+1} < 1$.*

Proof. By (7) we have

$$-Q(t+1) = (-1)^t \sum_{i=1}^{t} \binom{t}{i-1} d^i = (-1)^t d \sum_{j=0}^{t-1} \binom{t}{j} d^j = (-1)^t d \big((1+d)^t - d^t \big).$$

Thus we get

$$\theta_{t+1} = c^{t+1}Q(t+1) = d^{-t}\big((d+1)^t - d^t\big) = \left(1+\frac{1}{d}\right)^t - 1 = \frac{1}{q^t} - 1. \qquad (8)$$

Using $0 < q^{-t} < 2$ we have $0 < \theta_{t+1} < 1$. □

Claim 3. *Let $p < \frac{1}{t+1}$. Then $|\theta_Y| > |\theta_{Y+1}|$ for $Y = t+1, \dots, n-1$.*

Proof. By (7) we have $|Q(Y)| = \sum_{i=1}^t \phi(Y,i)$, where

$$\phi(Y,i) := \frac{(Y-1)\cdots(Y-t)}{(i-1)!(t-i)!(Y-i)} d^i > 0.$$

We prove $|\theta_Y| > |\theta_{Y+1}|$, or equivalently, $|c^Y Q(Y)| > |c^{Y+1}Q(Y+1)|$, by showing

$$|c^Y\phi(Y,i)|/|c^{Y+1}\phi(Y+1,i)| = d\,\phi(Y,i)/\phi(Y+1,i) > 1$$

for all $1 \le i \le t$. In fact we have

$$\frac{d\,\phi(Y,i)}{\phi(Y+1,i)} = \frac{(Y-t)(Y+1-i)}{Y(Y-i)}d \ge \frac{(Y-t)(Y+1-1)}{Y(Y-1)}d = \frac{Y-t}{Y-1}\,\frac{q}{p}.$$

If $t=1$ then the RHS is $q/p > 1$. If $t \ge 2$ then it follows from $Y \ge t+1$ that

$$\frac{Y-t}{Y-1}\,\frac{q}{p} \ge \frac{(t+1)-t}{(t+1)-1}\,\frac{q}{p} = \frac{q}{tp}.$$

Then $\frac{q}{tp} > 1$ is equivalent to our assumption $p < \frac{1}{t+1}$. □

The above two claims show (6). Then (4) and (6) give (Q2), and this completes the proof of Lemma 1. □

For those who might wonder how Lemma 1 relates to the Hoffman's ratio bound, we give a short remark here. In our case θ_i's play the role of "eigenvalues." Then the the largest one is $\theta_0 = p^{-t} - 1$ and the least one is $\theta_1 = -1$. (This is true for $p \le \frac{1}{t+1}$. We needed $q^{-t} < 2$ only to guarantee that $\theta_{t+1} < 1$.) Thus the corresponding ratio bound is $\frac{-\theta_1}{\theta_0 - \theta_1} = p^t$, as expected.

It follows from (3) with Lemma 1 that

$$\alpha_1\alpha_2\theta_0 = \sum_{x\neq\emptyset} \theta_{|x|}\hat{f}_1(x)\hat{f}_2(x) \le \sum_{x\neq\emptyset} |\hat{f}_1(x)\hat{f}_2(x)|. \qquad (9)$$

Applying the Cauchy–Schwarz inequality, and then by (C1) and (C2), we have

$$\sum_{x\neq\emptyset} |\hat{f}_1(x)\hat{f}_2(x)| \le \prod_{i=1}^2 \big(\sum_{x\neq\emptyset} \hat{f}_i(x)^2\big)^{\frac{1}{2}} = \prod_{i=1}^2 \sqrt{\alpha_i - \alpha_i^2} = \sqrt{\alpha_1\alpha_2}\sqrt{1-\alpha_1}\sqrt{1-\alpha_2}. \qquad (10)$$

Claim 4. *If $\alpha_1, \alpha_2 \in [0,1]$ then $\sqrt{1-\alpha_1}\sqrt{1-\alpha_2} \le 1 - \sqrt{\alpha_1\alpha_2}$.*

Proof. We apply the inequality of arithmetic and geometric means twice:

$$\sqrt{1-\alpha_1}\sqrt{1-\alpha_2} \le \frac{(1-\alpha_1)+(1-\alpha_2)}{2} = 1 - \frac{\alpha_1+\alpha_2}{2} \le 1 - \sqrt{\alpha_1\alpha_2}.$$

□

By (9), (10) and Claim 4 we have

$$\alpha_1\alpha_2\theta_0 \le \sum_{x\neq\emptyset} |\hat f_1(x)\hat f_2(x)| \le \sqrt{\alpha_1\alpha_2}\sqrt{1-\alpha_1}\sqrt{1-\alpha_2} \le \sqrt{\alpha_1\alpha_2}(1-\sqrt{\alpha_1\alpha_2}). \quad (11)$$

Using $\theta_0 = p^{-t}-1$ and rearranging we get $\alpha_1\alpha_2 \le p^{2t}$.

In the case of equality, we have equality in Claim 4. This gives $\alpha_1=\alpha_2$, that is, $\hat f_1(\emptyset)=\hat f_2(\emptyset)$ by (C1). We also have equality in (10), which gives

$$|\hat f_1(x)| = |\hat f_2(x)| \quad (12)$$

for all $x\neq\emptyset$. Moreover by the equality in (9) we have

$$\alpha_1\alpha_2\theta_0 = -\sum_{x\neq\emptyset}\theta_{|x|}\hat f_1(x)\hat f_2(x) = \sum_{x\neq\emptyset}|\hat f_1(x)\hat f_2(x)|.$$

This means that if $x\neq\emptyset$, then we have (i) $\hat f_1(x)=\hat f_2(x)=0$, or (ii) $\theta_{|x|}=-1$ and $\hat f_1(x)\hat f_2(x) = |\hat f_1(x)\hat f_2(x)|$. If (ii) happens, then it follows $\hat f_1(x)=\hat f_2(x)$ from (12). Consequently we have $\hat f_1(x)=\hat f_2(x)$ for all $x\in V$. Thus $f_1\equiv f_2$, that is, $\mathcal F_1=\mathcal F_2$. In this case, $\mathcal F_1$ is t-intersecting itself and $\mu_p(\mathcal F_1)=p^t$. Then $\mathcal F_1\cong\mathcal A_0(n,t)$ follows from Theorem 2. This completes the proof of Theorem 3.

3. Stability: Proof of Theorem 5

Let $\epsilon_0>0$ be a small absolute constant (independent from t and p) which is chosen so that ϵ_0 satisfies several inequalities appeared in this section. (One can easily verify that these inequalities hold by choosing ϵ_0 sufficiently small.) By taking c large enough so that $c\sqrt{\epsilon_0}\ge 1$, the theorem clearly holds for all $\epsilon\ge\epsilon_0$. Thus it suffices to show that the theorem holds for all $0<\epsilon<\epsilon_0$.

We say that a family of subsets $\mathcal F\subset 2^{[n]}$ is an upset if $G\supset F\in\mathcal F$ implies $G\in\mathcal F$. Without loss of generality we may assume that both $\mathcal F_1$ and $\mathcal F_2$ are upsets. To see this, suppose that the theorem is true for cross t-intersecting upsets. If $\mathcal F_1$ and $\mathcal F_2$ are not necessarily upsets, but they are cross t-intersecting families with

$$\mu_p(\mathcal F_1)\mu_p(\mathcal F_2) > (1-\epsilon)^2p^{2t},$$

then we can find upsets $\tilde{\mathcal F}_1\supset\mathcal F_1$ and $\tilde{\mathcal F}_2\supset\mathcal F_2$. By the assumption we have

$$\mu_p(\tilde{\mathcal F}_i \bigtriangleup \mathcal G) < \tilde c\sqrt\epsilon$$

for some $\mathcal G\cong\mathcal A_0(n,t)$ and $i=1,2$. Let $\mu_p(\tilde{\mathcal F}_i\setminus\mathcal F)=\epsilon_i$ and suppose that $\epsilon_1\ge\epsilon_2$. We claim that $\epsilon_1<\sqrt\epsilon$, which will give

$$\mu_p(\mathcal F_i\bigtriangleup\mathcal G)\le \epsilon_i+\mu_p(\tilde{\mathcal F}_i\bigtriangleup\mathcal F) < (1+\tilde c)\sqrt\epsilon.$$

Namely, by setting $c=1+\tilde c$, the theorem is true for $\mathcal F_1$ and $\mathcal F_2$ as well. To this end we use Theorem 3 to get

$$\begin{aligned} p^{2t} &\ge \mu_p(\tilde{\mathcal F}_1)\mu_p(\tilde{\mathcal F}_2) = (\mu_p(\mathcal F_1)+\epsilon_1)(\mu_p(\tilde{\mathcal F}_2)+\epsilon_2)\\ &\ge (\mu_p(\mathcal F_1)+\epsilon_1)\mu_p(\tilde{\mathcal F}_2) > (\mu_p(\mathcal F_1)+\epsilon_1)\frac{(1-\epsilon)^2p^{2t}}{\mu_p(\mathcal F_1)}. \end{aligned}$$

This gives

$$\epsilon_1 < \frac{1-(1-\epsilon)^2}{(1-\epsilon)^2}\mu_p(\mathcal{F}_1) \le \frac{2\epsilon-\epsilon^2}{(1-\epsilon)^2}.$$

The RHS is less than $\sqrt{\epsilon}$ for $\epsilon < 0.15$, as needed. (Thus we need to choose $\epsilon_0 < 0.15$ and this is one of the constraints for ϵ_0.) Therefore we may assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are upsets from the beginning.

Before starting the actual proof we briefly explain our plan. We use notation in the proof of Theorem 3. We fix t and p, and treat them as constants. Define the norm of $f \in \mathcal{H}_n$ by

$$|f|_2 := \sqrt{\langle f, f\rangle_p}.$$

Let $\mathbf{B}_n := \{f : 2^{[n]} \to \{0,1\}\} \subset \mathcal{H}_n$ be the set of characteristic functions. Let $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ and suppose that they are cross t-intersecting upsets. For $i = 1, 2$, let $f_i \in \mathbf{B}_n$ be the characteristic function of $\mathcal{F}_i$. Then we can write

$$\alpha_i := \mu_p(\mathcal{F}_i) = |f_i|_2^2,$$

and it follows from Theorem 3 that $|f_1|_2^2|f_2|_2^2 \le p^{2t}$. By symmetry we may assume that $|f_2|_2^2 \le |f_1|_2^2$. Then we have

$$|f_2|_2^2 \le p^t.$$

On the other hand, using our assumption

$$\sqrt{\alpha_1\alpha_2} = |f_1|_2|f_2|_2 > (1-\epsilon)p^t, \tag{13}$$

we will show that

$$\sum_{|x|>t} \hat{f}_2(x)^2 = O(\sqrt{\epsilon}),$$

namely, the LHS is less than $C\sqrt{\epsilon}$ for some constant C and all $0 < \epsilon < \epsilon_0$. Then we can apply the following result due to Friedgut [10]. (Lemma 2.8 in [10] is stated slightly differently, but what is actually proved there is exactly as follows.)

Theorem 9 ([10])**.** *Let $t \ge 1$ be an integer and let $0 < p < 1/2$. Let $f \in \mathbf{B}_n$ be a characteristic function of some (not necessarily t-intersecting) upset. If $\sum_{|x|>t} \hat{f}(x)^2 < \tilde{\epsilon}$ and $|f|_2^2 \le p^t$, then either $|f|_2^2 = O(\tilde{\epsilon})$, or there is a t-intersecting family $\mathcal{G} \cong \mathcal{A}_0(n,t)$ with the characteristic function $g \in \mathbf{B}_n$ such that $|f-g|_2^2 = O(\tilde{\epsilon})$.*

Theorem 9 will imply that there is a family $\mathcal{G} \cong \mathcal{A}_0(n,t)$ with the characteristic function g such that

$$|f_2 - g|_2^2 = \mu_p(\mathcal{F}_2 \triangle \mathcal{G}) = O(\sqrt{\epsilon}).$$

Next we will show that $|f_1 - f_2|_2^2 = O(\sqrt{\epsilon})$, which will imply

$$\mu_p(\mathcal{F}_1 \triangle \mathcal{G}) = |f_1 - g|_2^2 = |(f_1 - f_2) + (f_2 - g)|_2^2 \le (|f_1 - f_2|_2 + |f_2 - g|_2)^2 = O(\sqrt{\epsilon}).$$

This is the plan of our proof.

Now we get into details. By (4) and (8) we find a "spectral gap"

$$\delta := |\theta_t| - |\theta_{t+1}| = 1 - (q^{-t} - 1) = 2 - q^{-t} > 0.$$

Notice δ is a constants depending only on t and p. We recall from (4) and (6) that

$$|\theta_1| = \cdots = |\theta_t| = 1 > 1 - \delta = |\theta_{t+1}| > \cdots > |\theta_n|. \tag{14}$$

For $i = 1, 2$, let $\beta_i := \sum_{x \neq \emptyset} \hat{f}_i(x)^2 = \alpha_i - \alpha_i^2 \in (0, 1)$. We define $\tau_i \in [0, 1]$ to divide β_i into two parts as follows:

$$\sum_{|x|>t} \hat{f}_i(x)^2 = \tau_i \beta_i, \qquad \sum_{1 \leq |x| \leq t} \hat{f}_i(x)^2 = (1 - \tau_i)\beta_i.$$

We will show that both τ_1 and τ_2 are small. First we check that the product $\tau_1 \tau_2$ is small.

Claim 5. $\sqrt{\tau_1 \tau_2} = O(\epsilon)$.

Proof. By (9) and (14) we get

$$\alpha_1 \alpha_2 \theta_0 \leq (1 - \delta) \sum_{|x|>t} |\hat{f}_1(x) \hat{f}_2(x)| + \sum_{1 \leq |x| \leq t} |\hat{f}_1(x) \hat{f}_2(x)|.$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \alpha_1 \alpha_2 \theta_0 &\leq (1 - \delta) \sqrt{\tau_1 \beta_1} \sqrt{\tau_2 \beta_2} + \sqrt{(1 - \tau_1)\beta_1} \sqrt{(1 - \tau_2)\beta_2} \\ &= \sqrt{\beta_1 \beta_2} \left((1 - \delta) \sqrt{\tau_1 \tau_2} + \sqrt{(1 - \tau_1)(1 - \tau_2)} \right). \end{aligned} \tag{15}$$

Using Claim 4 we have

$$\sqrt{\beta_1 \beta_2} = \sqrt{\alpha_1 \alpha_2} \sqrt{1 - \alpha_1} \sqrt{1 - \alpha_2} \leq \sqrt{\alpha_1 \alpha_2}(1 - \sqrt{\alpha_1 \alpha_2})$$

and

$$\sqrt{1 - \tau_1} \sqrt{1 - \tau_2} \leq 1 - \sqrt{\tau_1 \tau_2}.$$

Thus we get

$$\alpha_1 \alpha_2 \theta_0 \leq \sqrt{\alpha_1 \alpha_2}(1 - \sqrt{\alpha_1 \alpha_2}) \left((1 - \delta) \sqrt{\tau_1 \tau_2} + (1 - \sqrt{\tau_1 \tau_2}) \right),$$

that is,

$$\sqrt{\alpha_1 \alpha_2}\, \theta_0 \leq (1 - \sqrt{\alpha_1 \alpha_2})(1 - \delta \sqrt{\tau_1 \tau_2}).$$

This gives

$$\sqrt{\alpha_1 \alpha_2} \leq \frac{1 - \delta \sqrt{\tau_1 \tau_2}}{1 + \theta_0 - \delta \sqrt{\tau_1 \tau_2}}.$$

Since $\sqrt{\alpha_1 \alpha_2} > (1 - \epsilon) p^t$ the above inequality implies

$$\sqrt{\tau_1 \tau_2} < \frac{\epsilon}{\delta(1 - (1 - \epsilon) p^t)} < \frac{\epsilon}{\delta(1 - p^t)} = O(\epsilon), \tag{16}$$

as desired. □

To show that τ_1 and τ_2 are close to each other we need a stronger version of Claim 4 as follows.

Claim 6. *Let $a_1, a_2, \gamma \in [0, 1]$. If $a_1 \geq a_2 + 8\gamma$ then*

$$\sqrt{1 - a_1} \sqrt{1 - a_2} \leq 1 - \sqrt{a_1 a_2} - 2\gamma^2.$$

Proof. We start with noting that

$$\sqrt{1-a_2} \geq \sqrt{1-a_1+8\gamma} \geq \sqrt{1-a_1}+2\gamma. \tag{17}$$

In fact the second inequality follows from

$$1-a_1+8\gamma \geq (1-a_1)+4\gamma^2+4\gamma\sqrt{1-a_1},$$

or equivalently, $2 \geq \gamma + \sqrt{1-a_1}$, which clearly holds.

For $b_1, b_2, \xi \in \mathbb{R}$ if $b_2 - b_1 \geq 2\xi$ then

$$\frac{b_1^2+b_2^2}{2} - b_1 b_2 = \frac{(b_2-b_1)^2}{2} \geq \frac{(2\xi)^2}{2} = 2\xi^2,$$

that is, $b_1 b_2 \leq \frac{b_1^2+b_2^2}{2} - 2\xi^2$. To apply this inequality, let $b_i := \sqrt{1-a_i}$ and $\xi := \gamma$. Then we have $b_2 - b_1 \geq 2\gamma$ from (17), and thus

$$\sqrt{1-a_1}\sqrt{1-a_2} \leq \frac{(1-a_1)+(1-a_2)}{2} - 2\gamma^2 = 1 - \frac{a_1+a_2}{2} - 2\gamma^2 \leq 1 - \sqrt{a_1 a_2} - 2\gamma^2.$$

□

Claim 7. $\max\{\tau_1, \tau_2\} = O(\sqrt{\epsilon})$.

Proof. Suppose that $\tau_1 \geq \tau_2$. (The opposite case can be proved in the same way.) First we show that

$$\tau_1 - \tau_2 < 8\gamma, \tag{18}$$

where $\gamma = \sqrt{\frac{\epsilon}{2(1-(1-\epsilon)p^t)}}$. Suppose, to the contrary, that $\tau_1 - \tau_2 \geq 8\gamma$. Then by Claim 6 we have

$$\sqrt{1-\tau_1}\sqrt{1-\tau_2} \leq 1 - \sqrt{\tau_1\tau_2} - 2\gamma^2.$$

Using this with (15) we get

$$\begin{aligned} \alpha_1\alpha_2\theta_0 &\leq \sqrt{\beta_1\beta_2}\big((1-\delta)\sqrt{\tau_1\tau_2} + (1-\sqrt{\tau_1\tau_2}-2\gamma^2)\big) \\ &\leq \sqrt{\alpha_1\alpha_2}(1-\sqrt{\alpha_1\alpha_2})(1-2\gamma^2-\delta\sqrt{\tau_1\tau_2}), \end{aligned}$$

and so

$$\sqrt{\alpha_1\alpha_2} \leq \frac{1-2\gamma^2-\delta\sqrt{\tau_1\tau_2}}{\theta_0+1-2\gamma^2-\delta\sqrt{\tau_1\tau_2}} < \frac{1-2\gamma^2}{\theta_0+1-2\gamma^2} = (1-\epsilon)p^t,$$

which contradicts (13). This proves (18).

It follows from $\tau_2 \leq \tau_1$ and (16) that

$$\tau_2 \leq \sqrt{\tau_1\tau_2} < \frac{\epsilon}{\delta(1-p^t)}.$$

This and (18) give

$$\tau_1 < \tau_2 + 8\gamma < \frac{\epsilon}{\delta(1-p^t)} + \sqrt{\frac{32\epsilon}{1-p^t}} = O(\sqrt{\epsilon}).$$

□

Without loss of generality we may assume that $|f_2|_2^2 \leq |f_1|_2^2$. Since $|f_1|_2^2|f_2|_2^2 \leq p^{2t}$ we have $|f_2|_2^2 \leq p^t$. In other words, we have $\alpha_1 \geq \alpha_2$, $\alpha_2 \leq p^t$. Recall from (13) that $\sqrt{\alpha_1\alpha_2} > (1-\epsilon)p^t$.

Claim 8. $\alpha_1 - \alpha_2 < 4\sqrt{2\epsilon}$.

Proof. Suppose, to the contrary, that $\alpha_1 - \alpha_2 \geq 8\gamma$ where $\gamma = \sqrt{\epsilon/2}$. Then by Claim 6 we have

$$\sqrt{1-\alpha_1}\sqrt{1-\alpha_2} \leq 1 - \sqrt{\alpha_1\alpha_2} - 2\gamma^2.$$

Thus by (11) we get

$$\alpha_1\alpha_2\theta_0 \leq \sqrt{\alpha_1\alpha_2}\sqrt{1-\alpha_1}\sqrt{1-\alpha_2} \leq \sqrt{\alpha_1\alpha_2}(1 - \sqrt{\alpha_1\alpha_2} - 2\gamma^2),$$

or $\sqrt{\alpha_1\alpha_2} \leq p^t(1-2\gamma^2) = (1-\epsilon)p^t$, which is a contradiction. □

Claim 9. $\alpha_2 > (1-\epsilon)p^t - 4\sqrt{2\epsilon}$.

Proof. Using Claim 8 we have

$$\big((1-\epsilon)p^t\big)^2 < \alpha_1\alpha_2 < (\alpha_2 + 4\sqrt{2\epsilon})\alpha_2 < (\alpha_2 + 4\sqrt{2\epsilon})^2,$$

and $(1-\epsilon)p^t < \alpha_2 + 4\sqrt{2\epsilon}$. □

We are going to apply Theorem 9 to f_2. By Claim 7 we have

$$\sum_{|x|>t} \hat{f}_2(x)^2 = \tau_2\beta_2 < \tau_2 = O(\sqrt{\epsilon}).$$

Recall that $\alpha_2 = |f_2|_2^2 \leq p^t$. On the other hand, if $|f_2|_2^2 = O(\sqrt{\epsilon})$, then this contradicts Claim 9. Thus $|f_2|_2^2 = O(\sqrt{\epsilon})$ is not the case, and then Theorem 9 gives

$$\mu_p(\mathcal{F}_2 \triangle \mathcal{G}) = |f_2 - g|_2^2 = O(\sqrt{\epsilon}) \tag{19}$$

for some family $\mathcal{G} \cong \mathcal{A}_0(n,t)$ with the characteristic function $g \in \mathbf{B}_n$.

Now we will show that $|f_1 - f_2|_2^2 = O(\sqrt{\epsilon})$ to prove $\mu_p(\mathcal{F}_1 \triangle \mathcal{G}) = O(\sqrt{\epsilon})$.

Claim 10. $\alpha_1 + \alpha_2 \leq 2p^t + 4\sqrt{2\epsilon}$.

Proof. This follows from Claim 8 and $\alpha_2 \leq p^t$. □

Claim 11. $\sum_{x\neq\emptyset} \hat{f}_1(x)\hat{f}_2(x) = \alpha_1\alpha_2\theta_0 + O(\epsilon)$.

Proof. Claim 5 gives

$$\sum_{|x|>t} \hat{f}_1(x)\hat{f}_2(x) \leq \sum_{|x|>t} |\hat{f}_1(x)\hat{f}_2(x)| \leq \sqrt{\tau_1\beta_1}\sqrt{\tau_2\beta_2} < \sqrt{\tau_1\tau_2} = O(\epsilon). \tag{20}$$

By (9) and (4) we have

$$\sum_{1\leq|x|\leq t} \hat{f}_1(x)\hat{f}_2(x) = \alpha_1\alpha_2\theta_0 + \gamma, \tag{21}$$

where $\gamma = \sum_{|x|>t} \theta_{|x|}\hat{f}_1(x)\hat{f}_2(x)$. By (14) and (20) we have

$$|\gamma| \leq (1-\delta)\sum_{|x|>t} |\hat{f}_1(x)\hat{f}_2(x)| = O(\epsilon). \tag{22}$$

Then the the desired result follows from (20), (21) and (22). □

Claim 12. $|f_1 - f_2|_2^2 = O(\sqrt{\epsilon})$.

Proof. We have

$$\begin{aligned}|f_1 - f_2|_2^2 &= \langle f_1 - f_2, f_1 - f_2\rangle = \langle f_1, f_1\rangle + \langle f_2, f_2\rangle - 2\langle f_1, f_2\rangle \\ &= \sum_{x\neq\emptyset} \hat{f}_1(x)^2 + \sum_{x\neq\emptyset} \hat{f}_2(x)^2 - 2\sum_{x\neq\emptyset} \hat{f}_1(x)\hat{f}_2(x) + (1+1-2) \\ &= (\alpha_1 - \alpha_1^2) + (\alpha_2 - \alpha_2^2) - 2\sum_{x\neq\emptyset} \hat{f}_1(x)\hat{f}_2(x) \quad \text{(by Claim 1 (C1) and (C2))} \\ &\le (\alpha_1 + \alpha_2) - 2\alpha_1\alpha_2 - 2\sum_{x\neq\emptyset} \hat{f}_1(x)\hat{f}_2(x) \quad \text{(by AM–GM ineq.)} \\ &= (\alpha_1 + \alpha_2) - 2\alpha_1\alpha_2(1+\theta_0) + O(\epsilon). \quad \text{(by Claim 11)}\end{aligned}$$

Then we use Claim 10 to bound the first term, and we use (13) and $1 + \theta_0 = p^{-t}$ for the second term. Thus we get

$$|f_1 - f_2|_2^2 < (2p^t + 4\sqrt{2\epsilon}) - 2((1-\epsilon)p^t)^2\, p^{-t} + O(\epsilon) = O(\sqrt{\epsilon}).$$

□

By (19) and Claim 12 we have

$$\mu_p(\mathcal{F}_1 \triangle \mathcal{G}) = |f_1 - g|_2^2 = |(f_1 - f_2) + (f_2 - g)|_2^2 \le (|f_1 - f_2|_2 + |f_2 - g|_2)^2 = O(\sqrt{\epsilon}),$$

which completes the proof of Theorem 5.

4. Integer sequences: Proof of Theorem 7 and Theorem 8

For $H \subset [m]^n$, $1 \le j \le n$ and $c \in [m]$, we define a shifting operation $S_{j,c}(H) = \{S_{j,c}(a) : a \in H\} \subset [m]^n$ as follows. For $a = (a_1, \ldots, a_n)$ let $S_j(a_1, \ldots, a_n) := (b_1, \ldots, b_n)$ where $b_\ell = a_\ell$ for $\ell \in [n] \setminus \{j\}$ and $b_j = 1$, and let $S_{j,c}(a) = S_j(a)$ if $a_j = c$ and $S_j(a) \notin H$, otherwise let $S_{j,c}(a) = a$. Namely, by $S_{j,c}(a)$, we replace a_j with 1 if $a_j = c$, but we do this replacement only if the resulting sequence is not in the original set H. The following observation is due to Kleitman [12].

Claim 13. *Let $H_1, H_2 \subset [m]^n$, $1 \le j \le n$ and $c \in [m]$. If H_1 and H_2 are cross t-intersecting, then $S_{j,c}(H_1)$ and $S_{j,c}(H_2)$ are also cross t-intersecting.*

Starting from cross t-intersecting sets of sequences $H_1, H_2 \subset [m]^n$, we repeat the shifting operations simultaneously. Then after finitely many steps we eventually get "shifted" sets H_1' and H_2', namely, $S_{j,c}(H_1') = H_1'$ and $S_{j,c}(H_2') = H_2'$ for all j and c. Since $|H_1'| = |H_1|$ and $|H_2'| = |H_2|$, and we are interested in the maximum of $|H_1||H_2|$, we may assume that both H_1 and H_2 are shifted cross t-intersecting from the beginning.

Now we reduce the problem of cross t-intersecting sets of sequences to the problem of cross t-intersecting families of subsets with weights. For $H \subset [m]^n$ and $a = (a_1, \ldots, a_n) \in H$, let $\sigma(a) := \{i : a_i = 1\} \subset [n]$ and $\sigma(H) := \{\sigma(a) : a \in H\} \subset 2^{[n]}$. Frankl and Füredi [7] observed the following.

Claim 14. *Let $H_1, H_2 \subset [m]^n$. If H_1 and H_2 are shifted cross t-intersecting sets of sequences, then $\sigma(H_1)$ and $\sigma(H_2)$ are cross t-intersecting families of subsets.*

Proof. For $\sigma(a) \in \sigma(H_1)$ and $\sigma(b) \in \sigma(H_2)$ we need to show that $|\sigma(a) \cap \sigma(b)| \geq t$. Let $a = (a_1, \ldots, a_n) \in H_1$ and $b = (b_1, \ldots, b_n) \in H_2$, and let $I := \{i : a_i = b_i \neq 1\}$. Since H_1 is shifted, we have $(a'_1, \ldots, a'_n) \in H_1$, where $a'_j = a_j$ if $j \notin I$, and $a'_i = 1$ if $i \in I$. Then cross t-intersecting property implies that

$$\begin{aligned} t &\leq \#\{j \in [n] : a'_j = b_j\} = \#\{j \in [n] \setminus I : a'_j = b_j\} \\ &= \#\{j \in [n] : a_j = b_j = 1\} = |\sigma(a) \cap \sigma(b)|. \end{aligned}$$

□

Let $\tilde{f}_2(n, m, t) = \max\{m^{2n} \mu_{\frac{1}{m}}(\mathcal{F}_1) \mu_{\frac{1}{m}}(\mathcal{F}_2)\}$, where $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ run over all cross t-intersecting families.

Proof of Theorem 7. If $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ are cross t-intersecting, then by letting $H_i := \{a \in [m]^n : \sigma(a) \in \mathcal{F}_i\}$ for $i = 1, 2$, we have

$$|H_i| = \sum_{x \in \mathcal{F}_i} (m-1)^{n-|x|} = m^n \sum_{x \in \mathcal{F}_i} (\tfrac{1}{m})^{|x|} (1 - \tfrac{1}{m})^{n-|x|} = m^n \mu_{\frac{1}{m}}(\mathcal{F}_i),$$

and H_1 and H_2 are clearly cross t-intersecting sets of integer sequences. This gives $f_2(n, m, t) \geq \tilde{f}_2(n, m, t)$.

On the other hand, if $H_1, H_2 \subset [m]^n$ are cross t-intersecting sets of integer sequences, then by letting $\mathcal{F}_i := \sigma(H_i) \subset 2^{[n]}$ for $i = 1, 2$, we have

$$|H_i| \leq \sum_{x \in \mathcal{F}_i} (m-1)^{n-|x|} = m^n \mu_{\frac{1}{m}}(\mathcal{F}_i),$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross t-intersecting by Claim 14. This gives $f_2(n, m, t) \leq \tilde{f}_2(n, m, t)$. □

Proof of Theorem 8. Let $p := 1/m$. If $m > (1 - \frac{1}{\sqrt[t]{2}})^{-1}$, or equivalently, $p < 1 - \frac{1}{\sqrt[t]{2}}$, then by Theorem 3 we have

$$\tilde{f}_2(n, m, t) = \max\{m^{2n} \mu_p(\mathcal{F}_1) \mu_p(\mathcal{F}_2)\} = m^{2n} p^{2t} = m^{2n} (1/m)^{2t} = (m^{n-t})^2.$$

Since $f_2(n, m, t) = \tilde{f}_2(n, m, t)$ by Theorem 7, we have $f_2(n, m, t) = (m^{n-t})^2$. □

Acknowledgment

The author thank the referee for valuable comments which improve the presentation of the paper.

References

[1] R. Ahlswede, L.H. Khachatrian. The diametric theorem in Hamming spaces — optimal anticodes. *Adv. in Appl. Math.*, 20:429–449, 1998.

[2] C. Bey, K. Engel. Old and new results for the weighted t-intersection problem via AK-methods. *Numbers, Information and Complexity, Althofer, Ingo, Eds. et al., Dordrecht*, Kluwer Academic Publishers, 45–74, 2000.

[3] I. Dinur, S. Safra. On the Hardness of Approximating Minimum Vertex-Cover. *Annals of Mathematics*, 162:439-485, 2005.

[4] D. Ellis, E. Friedgut, H. Pilpel. Intersecting families of permutations. *J. Amer. Math. Soc.*, 24 (2011) 649–682.

[5] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[6] P. Frankl, S. J. Lee, M. Siggers, N. Tokushige. An Erdős–Ko–Rado theorem for cross t-intersecting families. *preprint*, arXiv:1303.0657.

[7] P. Frankl, Z. Füredi. The Erdős–Ko–Rado theorem for integer sequences. *SIAM J. Alg. Disc. Math.*, 1:376–381, 1980.

[8] P. Frankl, N. Tokushige. The Erdős–Ko–Rado theorem for integer sequences. *Combinatorica*, 19 (1999) 55–63.

[9] P. Frankl, R. M. Wilson. The Erdős–Ko–Rado theorem for vector spaces. *J. Combin. Theory (A)*, 43 (1986) 228–236.

[10] E. Friedgut. On the measure of intersecting families, uniqueness and stability. *Combinatorica*, 28 (2008) 503–528.

[11] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.* 20 (2010) 416–526.

[12] D. J. Kleitman. On a combinatorial conjecture of Erdös. *J. of Combin. Theory*, 1 (1966) 209–213.

[13] M. Matsumoto, N. Tokushige. The exact bound in the Erdős–Ko–Rado theorem for cross-intersecting families. *J. of Combin. Theory (A)*, 22 (1989) 90–97.

[14] R.M. Wilson. The exact bound in the Erdős–Ko–Rado theorem. *Combinatorica*, 4:247–257, 1984.

[15] N. Tokushige. Intersecting families — uniform versus weighted. *Ryukyu Math. J.*, 18:89–103, 2005.

[16] N. Tokushige. The eigenvalue method for cross t-intersecting families. To appear in *J. Alg. Comb.* DOI: 10.1007/s10801-012-0419-4.

College of Education, Ryukyu University, Nishihara, Okinawa 903-0213, Japan
E-mail address: `hide@edu.u-ryukyu.ac.jp`