Every graph is an integral distance graph in the plane

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Abstract

We prove that every finite simple graph can be drawn in the plane so that any two vertices have an integral distance if and only if they are adjacent. The proof is constructive.

1 Introduction

In 1945, Anning and Erdős[1, 2] showed that for any n we can find n points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line). Then, Graham, Rothschild and Straus[3] showed that there are $d+2$ points in $\mathbf{R}^d$ whose distances are all odd integers if and only if $d \equiv 14 \pmod{16}$. In particular, four points in the plane with pairwise odd integral distances do not exist.

Let G be a simple graph. An injection $f : V(G) \to \mathbf{R}^2$ is said to be an integral (resp. rational) distance representation of G if the following condition holds:

$$xy \in E(G) \quad \text{if and only if} \quad |f(x) - f(y)| \in \mathbf{Z} \text{ (resp. } \mathbf{Q}).$$

In problems included in [5], Maehara asked whether there is a finite simple graph G which has no integral distance representation. In this paper, we prove the following.

Theorem 1 *Every finite simple graph has an integral distance representation (in the plane).*

Actually, we shall construct an integral distance representation on a circle with all rational distances.

What can we say about infinite graphs? On a circle, we can construct rational distance representations of the complete graph K_∞ (see the next section) and the complete bipartite graph $K_{\infty,\infty}$ (see [4]), where ∞ stands for $|\mathbf{Z}|$. On the other hand, Maehara[4] showed that $K_{3,\infty}$ has no integral distance representation. The proof is similar to the one in [1]. ($K_{2,\infty}$ has an integral distance representation.)

Problem 1 *Is there a (finite or infinite) simple graph G which has no rational distance representation?*

2 Proof of the theorem

1° Suppose that a quadrilateral $ABCD$ is inscribed in a circle. Then, by the Ptolemy theorem, we have

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

If five edges among the above six edges have rational lengths, then the length of the remaining edge is rational, too.

Let $\mathcal{S}$ be a semicircle with diameter $AB = 1$. Choose integers a and b so that $a > b > 0$. Then a point C on $\mathcal{S}$ satisfying $AC = \frac{2ab}{a^2+b^2}$ $BC = \frac{a^2-b^2}{a^2+b^2}$ is uniquely determined. (Note that $AC^2 + BC^2 = 1$.) We denote $<a + b\sqrt{-1}>$ to indicate the point C on $\mathcal{S}$. Define

$$\mathcal{F} := \{<z> \in \mathcal{S} : z \in \mathbf{Z}[\sqrt{-1}],\ 0 < \arg(z) < \pi/4\}.$$

Using the Ptolemy theorem, the distance d between two points in $\mathcal{F}$, $<a + b\sqrt{-1}>$ and $<s + t\sqrt{-1}>$, is given by

$$d = \frac{2 \mid ab(s^2 - t^2) - st(a^2 - b^2) \mid}{(a^2 + b^2)(s^2 + t^2)}. \qquad (1)$$

Since any two points in $\mathcal{F}$ has rational distance, we can find a rational distance representation of the complete graph of arbitrary order in $\mathcal{F}$.

2° Let K_n be the complete graph of order n, and set $V(K_n) = \{v_1 \dots, v_n\}$, $E(K_n) = \{e_1 \dots, e_m\}$, $m = \binom{n}{2}$. Let $q_1, \dots, q_m \in \mathbf{Z}^+$ be mutually prime.

Let $f = f_{(n;q_1,\dots,q_m)}$ be a rational distance representation of K_n satisfying

$$\text{(the length of } e_k \text{ in } f) = \frac{*}{q_k},$$

i.e., the denominator of the irreducible fraction of $|f(x)-f(y)|$ where $e_k = xy$ is precisely q_k for $k = 1, \dots, m$. Let $G = (V, E)$ be a given graph. Multiplying f by $\prod_{e_k \in E} q_k$, we have an integral distance representation of G. Thus, to prove the theorem, it suffices to find an $f_{(n;q_1,\dots,q_m)}$.

3° Suppose that $<z_1>, <z_2> \in \mathcal{F}$, and set $z_1 = a + b\sqrt{-1}$, $z_2 = s + t\sqrt{-1}$. A pair $(<z_1>, <z_2>)$ of two points in $\mathcal{F}$ is called "q-good" if the following conditions hold:

(i) $a^2 + b^2 \equiv s^2 + t^2 \equiv 0 \pmod{q}$.

(ii) $\gcd\{2bt, q\} = 1$.

(iii) $at - bs \not\equiv 0 \pmod{q}$.

Then, by using (1), the numerator of distance d of the pair satisfies

$$\begin{aligned} & ab(s^2 + t^2 - 2t^2) - st(a^2 + b^2 - 2b^2) \\ & \equiv \; -2abt^2 + 2stb^2 \; \equiv \; -2bt(at - bs) \; \not\equiv \; 0 \pmod{q}. \end{aligned}$$

Note that if a q-good pair of distance d satisfies $\ell d \in \mathbf{Z}$ then $q^2 | \ell$ must hold.

4° Let $m = \binom{n}{2}$ and $q_0, q_1, \dots, q_m \in \mathbf{Z}^+$ be mutually prime. Associate q_k to the edge $e_k \in E(K_n)$ for $k = 1, \dots, m$. Suppose that there are $<z_i> \in \mathcal{F}$, $i = 1, \dots, n$, such that

$$|z_i|^2 = q_0 \prod_{v_i \in e_k} q_k \tag{2}$$

and

$$\text{for each } e_k = v_i v_j,\ (<z_i>, <z_j>) \text{ is } q_k\text{-good.} \tag{3}$$

Then we can define a rational distance representation f of K_n by $f(v_i) = <z_i>$, and multiplying f by $q_0^2 q_1 \cdots q_m$, we get a rational distance representation $f_{(n;q_1,\dots,q_m)}$. Thus, to prove the theorem, it suffices to find q_ks and z_is satisfying (2) and (3).

5° Here let us recall some elementary properties of $\mathbf{Z}[\sqrt{-1}]$. Define

$$\begin{aligned} Z_1 &:= \{z \in \mathbf{Z}[\sqrt{-1}] : 0 < \arg(z) < \pi/4\}, \\ P_1 &:= \{z \in Z_1 : z \text{ is a Gaussian prime}\}. \end{aligned}$$

Lemma 1 *Let $z \in Z_1$, and ϕ, δ be reals with $0 < \phi < 2\pi$, $\delta > 0$. Then we can choose $\ell \in \mathbf{Z}^+$ such that $|\phi - \arg(z^\ell)| < \delta$.*

Proof. Set $\theta = \arg(z)$ in degree measure. Since $\tan\theta$ is rational and $\theta \neq 45°$, it is known that θ is irrational. Thus we can choose a desired ℓ. ∎

Lemma 2 *Let $\alpha_1, \ldots, \alpha_h$ be distinct elements in P_1, and $\ell_1, \ldots, \ell_h \in \mathbf{Z}^+$. Set $\alpha_1^{\ell_1} \cdots \alpha_h^{\ell_h} = a + b\sqrt{-1}$. Then $|\alpha_1|^2, \ldots, |\alpha_h|^2, a, b$ are mutually prime.*

Proof. Let $p_i = |\alpha_i|^2$. Suppose that p_i divides a. Since $p_1^{\ell_1} \cdots p_h^{\ell_h} = a^2 + b^2$, we must have $p_i | b$. Then $p_i = \alpha_i \overline{\alpha}_i$ divides $a + b\sqrt{-1}$. Thus we have $\overline{\alpha}_i | (\alpha_1^{\ell_1} \cdots \alpha_h^{\ell_h})$ and $\overline{\alpha}_i | \alpha_i$. This implies $|\alpha_i|^2 = 1$, a contradiction. ∎

6° Let $m = \binom{n}{2}$ and $\alpha_0, \alpha_1, \ldots, \alpha_m$ be distinct elements in P_1. By Lemma 1, we can choose β_0 to be a power of α_0 with $\arg(\beta_0) \approx \pi/8$. In the same reason, for $k = 1, \ldots, m$, we can choose β_k to be a power of α_k with $0 < \arg(\beta_k) < \epsilon$, where $n\epsilon < \pi/8$. Set $q_k := |\beta_k|^2$ for $k = 0, 1, \ldots, m$. Then $q_0, \ldots, q_k$ are mutually prime, prime powers.

Let T_n be a tournament of order n, and set $V(T_n) = \{v_1 \ldots, v_n\}$, $E(T_n) = \{\vec{e}_1 \ldots, \vec{e}_m\}$. Associate β_k to the directed edge $\vec{e}_k$ for $k = 1, \ldots, m$. For $v_i \in \vec{e}_k$, define

$$z_i := \beta_0 \prod_{v_i \in \vec{e}_k} \gamma_{ik}, \tag{4}$$

where

$$\gamma_{ik} := \begin{cases} \beta_i & \text{if } \vec{e}_k \text{ is outedge at } v_i, \\ \overline{\beta}_i & \text{if } \vec{e}_k \text{ is inedge at } v_i. \end{cases}$$

By the choice of $\arg(\beta_i)$, we have $0 < \arg(z_i) < \pi/4$, i.e., $<z_i> \in \mathcal{F}$ for each i. By the construction, z_is and p_ks satisfy (2).

Next let us check the condition (3). Let $\vec{e}_k = \overrightarrow{v_i v_j}$. By (2), we have $|z_i|^2 \equiv |z_j|^2 \equiv 0 \pmod{q_k}$, and (i) in **3°** is satisfied. Set $\beta_k = u + v\sqrt{-1}$, $z_i = \beta_k(x + y\sqrt{-1})$, $z_j = \overline{\beta}_k(X + Y\sqrt{-1})$. Then we have

$$\begin{aligned} z_i &= (u + v\sqrt{-1})(x + y\sqrt{-1}) = (ux - vy) + (uy + vx)\sqrt{-1}, \\ z_j &= (u - v\sqrt{-1})(X + Y\sqrt{-1}) = (uX + vY) + (uY - vX)\sqrt{-1}. \end{aligned}$$

By (4) and Lemma 2, $(uy+vx)(uY-vX)$ and q_k are mutually prime. Hence (ii) is satisfied.

Finally we check (iii). Set $M := (ux-vy)(uY-vX)-(uX+vY)(uy+vx)$. We shall prove $M \not\equiv 0 \pmod{q_k}$. Suppose, on the contrary, that $M \equiv 0$. By Lemma 2, u, v and $q_k = |\beta_k|^2 = u^2+v^2$ are mutually prime. Thus we have

$$\begin{aligned} 0 &\equiv (u^2-v^2)(xY-yX) - 2uv(xX+yY) \\ &\equiv -2v^2(xY-yX) - 2uv(xX+yY) \\ &\equiv v(xY-yX) - u(xX+yY) \end{aligned}$$

in module q_k. This implies $v^2(xY-yX)^2 \equiv u^2(xX+yY)^2$. Using $u^2 \equiv -v^2$, it follows that

$$\begin{aligned} 0 &\equiv u^2\{(xY-yX)^2 + (xX+yY)^2\} \\ &\equiv (x^2+y^2)(X^2+Y^2) \\ &\equiv |z_i|^2|z_j|^2/q_k^2 \end{aligned}$$

in modulo q_k, which is a contradiction. This completes the proof of the theorem.

References

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