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When does a planar bipartite framework admit a continuous deformation?

H. Maehara *, N. Tokushige

College of Education, Ryukyu University, Nishihara, Okinawa 903-0213, Japan

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Abstract

Let $K(X,Y)$ denote the bipartite framework in the plane that realizes the complete bipartite graph $K_{m,n}$ with partite sets X, Y; $|X| = m, |Y| = n$. We show that for $m \geqslant 3, n \geqslant 5$, $K(X,Y)$ admits a *continuous* deformation if and only if X lies on a line ℓ and Y lies on a line perpendicular to ℓ.

Keywords: Framework; Bipartite; Deformation

1. Introduction

A *framework* in the plane is a graph whose vertices are points in the plane and whose edges are line segments connecting two vertices. By a *motion* of a framework G in the plane, we mean a continuous movement of the vertices of G in the plane that preserves the length of every edge. If a motion of G changes the distance between a pair of nonadjacent vertices, then the motion is called a *continuous deformation* (or simply a *deformation*) of G. A framework in the plane is called *flexible* if it admits a deformation, otherwise, it is called *rigid*. It is known that if a framework admits a 'continuous' deformation, then it admits a 'smooth' deformation, see [1]. So, we may consider only 'smooth' deformations.

Throughout this paper, X, Y denote two, disjoint, nonempty, finite-sets in the plane. The cardinalities of X, Y are denoted by $|X|, |Y|$. Let $K(X,Y)$ denote the *bipartite framework*, that is, the complete bipartite graph with partite sets X and Y. Since a bipartite framework has no triangles, problems concerning its rigidity or flexibility are considered with some interest. It is known [2, 3] that for almost all pair of sets X, Y with $|X|, |Y| \geqslant 3$, $K(X,Y)$ is rigid. Then, exactly when does $K(X,Y)$ admit a deformation? We prove the following.

* Corresponding author.

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Theorem 1. *For $|X|\geqslant 3, |Y|\geqslant 5$, the bipartite framework $K(X,Y)$ admits a continuous deformation if and only if X lies on a line ℓ and Y lies on a line perpendicular to ℓ.*

The *if* part of this theorem is easy. To see this, suppose that $X=\{p_1,p_2,\ldots\}$ lies on the x-axis and $Y=\{q_1,q_2,\ldots\}$ lies on the y-axis, with no q_j on the origin. Then, we can put

$$p_i=(\sigma_i\sqrt{a_i+t},0),\quad i=1,2,\ldots,$$

$$q_j=(0,\tau_j\sqrt{b_j-t}),\quad j=1,2,\ldots,$$

where $\sigma_i,\ \tau_j=\pm 1$ and $a_i,\ t\geqslant 0,\ b_j>0$. Then the length of $\overline{p_iq_j}$ is equal to a_i+b_j, which is irrelevant to t. Hence by varying t, we can deform $K(X,Y)$.

The *only if* part of the theorem is Proposition 4 in Section 5. Actually, we prove that unless X lies on a line ℓ and Y lies on a line perpendicular to ℓ, there are $X'\subset X, Y'\subset Y,\ |X'|=|Y'|=3$ such that $K(X',Y')$ is rigid.

Remark. Bottema gave an example of $K(X,Y)$, $|X|=|Y|=4$, that admits a continuous deformation, and neither X nor Y lies on a line, see e.g. [4] Thus, the above theorem does not hold for $|X|=|Y|=4$. We include the example in the last section for the self-completeness.

Problem 1. Characterize the flexible representations of $K(3,3)$, $K(3,4)$, $K(4,4)$ in the plane.

2. Infinitesimal deformations

A *vector field* f on $X\subset\mathbf{R}^2$ is a map $f:X\to\mathbf{R}^2$. When we want to show the domain of f explicitly, we use the notation $f|X$. If the values of f are obtained as the velocity vectors of a smooth rigid motion of the whole plane, then f is called *trivial*. The set of all vector fields on X naturally constitute a vector space.

An *infinitesimal motion* of a framework G is a vector field f on the vertex set of G that satisfies

$$(p-q)\cdot(f(p)-f(q))=0$$

for all edges $\overline{pq}$ of G, where $\cdot$ denotes the inner product. A nontrivial infinitesimal motion of G is called an *infinitesimal deformation* of G. If G admits an infinitesimal deformation, then G is called *infinitesimally flexible*, otherwise, G is called *infinitesimally rigid.* If G admits a deformation, then the velocity vectors of the vertices at some instant give an infinitesimal deformation of G. Hence, if G is flexible, then it is infinitesimally flexible.

Lemma 1. *Suppose that $|X|,|Y|\geqslant 2$ and no three points of $X\cup Y$ are collinear. Let f,g be two infinitesimal motions of $K(X,Y)$. If the values of f and g coincide at two vertices in X, then they are the same infinitesimal motion.*

Proof. Let $X = \{x_1, x_2, \dots\}, Y = \{y_1, y_2, \dots\}$ and suppose that $f(x_1) = g(x_1)$, $f(x_2) = g(x_2)$. Then for any y_j, we have

$$(x_i - y_j) \cdot (f(x_i) - f(y_j)) = 0, \quad (x_i - y_j) \cdot (f(x_i) - g(y_j)) = 0 \quad (i = 1, 2),$$

from which, we have

$$(x_i - y_j) \cdot (f(y_j) - g(y_j)) = 0, \quad i = 1, 2.$$

Since $x_1 - y_j$ and $x_2 - y_j$ are linearly independent, we have $f(y_j) = g(y_j)$. Similarly we have $f(x_i) = g(x_i)$ for $i > 2$. □

Let f, g be two infinitesimal motions of G. If $f - \lambda g$ is trivial for some $\lambda \neq 0$, then f is said to be *equivalent* to g, and we write as $f \sim g$. Note that $\sim$ is an equivalence relation.

Lemma 2. *All nontrivial vector fields on a* 2*-point-set* $X = \{p, q\}$ *are equivalent.*

Proof. Let f be the vector field on X such that $f(p) = 0, f(q) = q - p$. Let us show that any nontrivial vector field g on X is equivalent to f. Let h be the trivial vector field such that $h(p) = h(q) = g(p)$, and let $g' = g - h$. Then $g' \sim g$ and $g'(p) = 0$. Since g' is nontrivial, $g'(q) \neq 0$ and $g'(q)$ is not orthogonal to $q - p$. Represent $g'(q)$ as

$$g'(q) = \lambda(q - p) + \boldsymbol{v},$$

where $\boldsymbol{v}$ is a vector orthogonal to $q - p$. Then $\lambda \neq 0$. Let $j|X$ be the vector field such that $j(p) = 0, j(q) = \boldsymbol{v}$. Then j is trivial. (It is a velocity vector field of a rotation around p.) Hence, $g \sim g' \sim g' - j \sim f$. □

The next lemma follows from Lemmas 1 and 2.

Lemma 3. *Suppose that* $|X|, |Y| \geqslant 2$ *and no three points of* $X \cup Y$ *lie on a line. If* $K(X, Y)$ *is infinitesimally flexible, then all infinitesimal deformations of* $K(X, Y)$ *are equivalent.*

Lemma 4. *Let* $X = \{p_1, p_2, p_3\}$. *A non-trivial vector field* $f|X$ *can be extended to an infinitesimal motion of* $K(\{q\}, X), q \notin X$, *if and only if*

$$rank \begin{pmatrix} q - p_1 \\ q - p_2 \\ q - p_3 \end{pmatrix} = rank \begin{pmatrix} q - p_1 & (q - p_1) \cdot f(p_1) \\ q - p_2 & (q - p_2) \cdot f(p_2) \\ q - p_3 & (q - p_3) \cdot f(p_3) \end{pmatrix}.$$

Proof. Let $(x, y) = f(q)$. Then, since

$$(q - p_i) \cdot (f(q) - f(p_i)) = 0 \quad (i = 1, 2, 3)$$

$$\Leftrightarrow (q - p_i) \cdot f(q) = (q - p_i) \cdot f(p_i) \quad (i = 1, 2, 3)$$

$$\Leftrightarrow \begin{pmatrix} (q - p_1) \\ (q - p_2) \\ (q - p_3) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (q - p_1) \cdot f(p_1) \\ (q - p_2) \cdot f(p_2) \\ (q - p_3) \cdot f(p_3) \end{pmatrix},$$

the lemma follows from the theory of linear equations. □

Proposition 1. *Suppose that* $|X|, |Y| \geqslant 3$ *and no three of* $X \cup Y$ *are collinear. If* $K(X, Y)$ *is infinitesimally flexible, then* $X \cup Y$ *lies on a conic.*

Proof. Suppose that $K(X, Y)$ admits an infinitesimal deformation $f : X \cup Y \to \mathbf{R}^2$. Let $p_i = (a_i, b_i)$, $i = 1, 2, 3$, be some three vertices of X. Since $f|\{p_1, p_2, p_3\}$ can be extended to an infinitesimal deformation of $K(X, Y)$, it follows from Lemma 4 that for any vertex $q = (x, y) \in Y$,

$$\begin{vmatrix} x - a_1 & y - b_1 & (x - a_1, y - b_1) \cdot f(p_1) \\ x - a_2 & y - b_2 & (x - a_2, y - b_2) \cdot f(p_2) \\ x - a_3 & y - b_3 & (x - a_3, y - b_3) \cdot f(p_3) \end{vmatrix} = 0.$$

This gives a quadratic equation on x, y. Hence $\{p_1, p_2, p_3\} \cup Y$ lies on a conic. Similarly, for any $q_j \in X - \{p_1, p_2\}$, the set $\{p_1, p_2, p_j\} \cup Y$ lies on a conic. Since a 'proper' conic is determined by any five points on it, we can conclude that $X \cup Y$ lies on a conic. □

The following precise result is known [3].

Theorem 2 (Whiteley). *The planar bipartite framework* $K(X, Y)$, $|X| \geqslant 3, |Y| \geqslant 3$, *is infinitesimally flexible if and only if one of the following holds*:
(1) X *and a point of* Y *lie on a line.*
(2) Y *and a point of* X *lie on a line.*
(3) $X \cup Y$ *lie on a conic.*

3. Parabola

It follows from elementary linear algebra that six points (x_i, y_i), $i = 1, \ldots, 6$, lie on a conic if and only if

$$\begin{vmatrix} x_1^2 & y_1^2 & x_1 y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2 y_2 & x_2 & y_2 & 1 \\ & & \cdots & \cdots & & \\ x_6^2 & y_6^2 & x_6 y_6 & x_6 & y_6 & 1 \end{vmatrix} = 0.$$

Let us call the left-hand-side determinant the *conic-discriminant* of the six points.

Lemma 5. *Let* $(x_i(t), y_i(t))$, $i=1,2,\dots,6$, *be six points moving on the plane, and let* (a_i, b_i), $i=1,\dots,6$, *be their velocity vectors at* $t=0$. *Let*

$$\mathscr{D}(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_8 t^8$$

be the conic-discriminant of the uniformly moving six point

$$(x_i(0) + a_i t, y_i(0) + b_i t), \quad i = 1,\dots,6.$$

If for each $t \in [0,\varepsilon)$, *the six points* $(x_i(t), y_i(t))$, $i=1,2,\dots,6$, *lie on a conic, then* $\lambda_0 = \lambda_1 = 0$.

Proof. Let $\mathscr{F}(t)$ be the conic-discriminant of the six points $(x_i(t), y_i(t))$, $i=1,2,\dots,6$. Since $\mathscr{F}(0)=0$, we have $\lambda_0=0$. Let

$$\varepsilon_i(t) = x_i(t) - x_i(0) - a_i t, \quad \delta_i(t) = y_i(t) - y_i(0) - b_i t, \quad i = 1,\dots,6.$$

Then

$$\lim_{t\to 0} \frac{\varepsilon_i(t)}{t} = \lim_{t\to 0} \frac{\delta_i(t)}{t} = 0.$$

Hence, from the expansion of the determinant we have

$$\mathscr{F}(t) = \mathscr{D}(t) + \mathrm{o}(t) = \lambda_1 t + \cdots + \lambda_8 t^8 + \mathrm{o}(t).$$

Since $\mathscr{F}(t)=0$ for $0 \leqslant t < \varepsilon$, we get

$$0 = \lim_{t\to 0} \frac{\mathscr{F}(t)}{t} = \lambda_1. \qquad \square$$

Proposition 2. *Suppose that* $|X| \geqslant 3, |Y| \geqslant 5$ *and* $X \cup Y$ *lies on a parabola. Then* $K(X,Y)$ *admits no deformation.*

Proof. Suppose that $X \cup Y$ lies on a parabola $y = kx^2$ and that $K(X,Y)$ admits a deformation. Restricting the deformation within a small range, we may suppose that the vertices are always in general position, that is, no three vertices are collinear. Then, by Proposition 1, at any instant, the vertices of $K(X,Y)$ lie on a conic.

The infinitesimal deformation of $K(X,Y)$ is unique up to equivalence by Lemma 3. Hence, by superposing a certain smooth rigid motion and changing scale if necessary, we may suppose that the velocity vectors of the vertices are given by the assignment:

$$X \ni (x,y) \mapsto (kx, 0), \qquad Y \ni (x', y') \mapsto (-kx', 1). \tag{1}$$

Since $(x-x', y-y')\cdot(kx+kx', -1) = kx^2 - kx'^2 - y + y' = 0$, this assignment is indeed an infinitesimal deformation of $K(X,Y)$.

Let

$$(a, ka^2), (b, kb^2), (c, kc^2)$$

be the initial positions (at $t = 0$) of some three vertices in X and let

$$(u, ku^2), (v, kv^2), (w, kw^2)$$

be the initial positions of some three vertices in Y. Let $\mathscr{D}(t)$ be the conic-discriminant of the six points

$$(a, ka^2) + t(ka, 0), \quad (b, kb^2) + t(kb, 0), \quad (c, kc^2) + t(kc, 0),$$
$$(u, ku^2) + t(-ku, 1), \quad (v, kv^2) + t(-kv, 1), \quad (w, kw^2) + t(-kw, 1).$$

Then $\mathscr{D}(t)$ is

$$\begin{vmatrix} (1+kt)^2a^2 & k^2a^4 & (1+kt)ka^3 & (1+kt)a & ka^2 & 1 \\ (1+kt)^2b^2 & k^2b^4 & (1+kt)kb^3 & (1+kt)b & kb^2 & 1 \\ (1+kt)^2c^2 & k^2c^4 & (1+kt)kc^3 & (1+kt)c & kc^2 & 1 \\ (1-kt)^2u^2 & (ku^2+t)^2 & (1-kt)u(ku^2+t) & (1-kt)u & ku^2+t & 1 \\ (1-kt)^2v^2 & (kv^2+t)^2 & (1-kt)v(kv^2+t) & (1-kt)v & kv^2+t & 1 \\ (1-kt)^2w^2 & (kw^2+t)^2 & (1-kt)w(kw^2+t) & (1-kt)w & kw^2+t & 1 \end{vmatrix}.$$

We are going to show that for a certain choice of three vertices in Y, the coefficient λ_1 of t in $\mathscr{D}(t)$ is not zero. Then since the velocity vectors of the vertices are given by (1), and since the vertices of $K(X, Y)$ are, at any instant, lying on a conic, we have a contradiction by Lemma 5.

First, regarding λ_1 as a polynomial in w, let A be the coefficient of w^4. Next, regarding A as a polynomial in v, let B be the coefficient of v^3. Then B is the coefficient of t in the expansion of

$$\pm k^3(1-kt)\begin{vmatrix} (1+kt)^2a^2 & (1+kt)a & ka^2 & 1 \\ (1+kt)^2b^2 & (1+kt)b & kb^2 & 1 \\ (1+kt)^2c^2 & (1+kt)c & kc^2 & 1 \\ (1-kt)^2u^2 & (1-kt)u & ku^2+t & 1 \end{vmatrix},$$

which is equal to

$$\pm(a-b)(b-c)(c-a)(1+4k^2u^2)k^3.$$

Since a, b, c are all distinct, we have $B \neq 0$. Then, as a polynomial in v, $A = A(v)$ is not identically zero. Hence, the equation $A(v) = 0$ is a nontrivial equation on v with degree 3. Since $|Y| \geqslant 5$, we may suppose that $A \neq 0$ for our choice of v. (Note that if $v = u$, then $A = 0$. Hence $A \neq 0$ implies $v \neq u$.) Thus, as an equation on w, $\lambda_1 = 0$ is a nontrivial quatric equation. Since $|Y| \geqslant 5$, we may suppose that $\lambda_1 \neq 0$ for our choice of w. Thus, the coefficient of t in the expansion of $\mathscr{D}(t)$ is not zero, a contradiction.

□

4. Ellipse and hyperbola

Proposition 3. *Suppose that* $|X| \geqslant 3, |Y| \geqslant 5$ *and* $X \cup Y$ *lies on an ellipse or a hyperbola. Then* $K(X, Y)$ *admits no deformation.*

Proof. Proof goes along the similar line as the parabola case. The equation of an ellipse or hyperbola is given by $kx^2 + y^2 = 1$. If $k > 0$, it is an ellipse, and if $k < 0$, it is a hyperbola.

Suppose that $X \cup Y$ lie on a conic $kx^2 + y^2 = 1$ and that $K(X, Y)$ admits a deformation. By Lemma 3, we may suppose that the velocity vectors of the vertices are given by the following assignment:

$$X \ni (x, y) \mapsto (kx, y),$$
$$Y \ni (x', y') \mapsto -(kx', y').$$

Since $(x - x', y - y') \cdot (kx + kx', y + y') = kx^2 - kx'^2 + y^2 - y'^2 = 0$, this assignment is indeed an infinitesimal deformation of $K(X, Y)$.

Let

$$(a, \tau_1\sqrt{1 - ka^2}), \quad (b, \tau_2\sqrt{1 - kb^2}), \quad (c, \tau_3\sqrt{1 - kc^2}) \tag{2}$$

be the initial positions of some three vertices in X, and let

$$(u, \tau_4\sqrt{1 - ku^2}), \quad (v, \tau_5\sqrt{1 - kv^2}), \quad (w, \tau_6\sqrt{1 - kw^2}),$$

be the initial positions of some three vertices of Y, where $\tau_i = \pm 1$.

Let $\mathscr{D}(t)$ be the conic-discriminant of the six points

$$((1 + kt)a, \tau_1(1 + t)\sqrt{1 - ka^2}), \quad ((1 + kt)b, \tau_2(1 + t)\sqrt{1 - kb^2}),$$
$$((1 + kt)c, \tau_3(1 + t)\sqrt{1 - kc^2}), \quad ((1 - kt)u, \tau_4(1 - t)\sqrt{1 - ku^2}),$$
$$((1 - kt)v, \tau_5(1 - t)\sqrt{1 - kv^2}), \quad ((1 - kt)w, \tau_6(1 - t)\sqrt{1 - kw^2}).$$

To get a contradiction, we are going to show that the coefficient λ_1 of t in the expansion of $\mathscr{D}(t)$ is not zero.

Write λ_1 as

$$\lambda_1 = \tau_6 Aw\sqrt{1 - kw^2} + \tau_6 B\sqrt{1 - kw^2} + Cw^2 + Dw + E$$

and write A as

$$A = \tau_5 P\sqrt{1 - kv^2} + Qv^2 + Rv + S.$$

Then P is the coefficient of t in the expansion of

$$\pm(1 - kt)(1 - t)^2 \begin{vmatrix} (1 + kt)^2a^2 & (1 + t)^2(1 - ka^2) & (1 + kt)a & 1 \\ (1 + kt)^2b^2 & (1 + t)^2(1 - kb^2) & (1 + kt)b & 1 \\ (1 + kt)^2c^2 & (1 + t)^2(1 - kc^2) & (1 + kt)c & 1 \\ (1 - kt)^2u^2 & (1 - t)^2(1 - ku^2) & (1 - kt)u & 1 \end{vmatrix},$$

which is equal to

$$\pm 4(a-b)(b-c)(c-a)(1-ku^2+k^2u^2).$$

First, suppose that $(a-b)(b-c)(c-a)\neq 0$. Then, since $1-ku^2\geqslant 0$, we have $P\neq 0$, and the equation $A=0$ is a nontrivial equation on v. Here, remark that since $P\neq 0$, A changes its value according to $\tau_5=+1$ or $\tau_5=-1$, provided that $\sqrt{1-kv^2}\neq 0$. From the equation $A=0$, we get the quatric equation

$$P^2(1-kv^2)=(Qv^2+Rv+S)^2$$

on v. Since $|Y|\geqslant 5$ and in view of the above remark, we may suppose $A\neq 0$ for our choice of $(v,\tau_5\sqrt{1-kv^2})$. (Note that if $(v,\tau_5\sqrt{1-kv^2})=(u,\tau_4\sqrt{1-ku^2})$, then $A=0$. Hence $A\neq 0$ implies $(v,\tau_5\sqrt{1-kv^2})\neq(u,\tau_4\sqrt{1-ku^2})$.) Thus, as an equation on w, $\lambda_1=0$ is a nontrivial equation. Note also that λ_1 changes its value according to $\tau_6=+1$ or $\tau_6=-1$, unless $\sqrt{1-kw^2}=0$. The equation $\lambda_1=0$ yields the quatric equation

$$(Aw+B)^2(1-kw^2)=(Cw^2+Dw+E)^2$$

on w, and since $|Y|\geqslant 5$, we can choose $(w,\tau_6\sqrt{1-kv^2})$ so that $\lambda_1\neq 0$.

Next, suppose that $(a-b)(b-c)(c-a)=0$. Then, since the three points in (2) are all distinct, we may assume that

$$a=b\neq c,\quad \tau_1=-\tau_2. \tag{3}$$

Then, A can be computed as

$$\begin{aligned}A=&\,2(c-a)u(-a-c+u+acku-ack^2u)\\&+2(c-a)(a+c)(1-ku^2+k^2u^2)v\\&+2(c-a)(-1-ack+ack^2+aku+cku-ak^2u-ck^2u^2)v^2.\end{aligned}$$

If $a\neq -c$ then the coefficient of v is not zero. If $a=-c$, then $a\neq 0$ by (3), and

$$A=4a(-1+a^2k-a^2k^2)u^2+4a(1-a^2k+a^2k^2)v^2,$$

which is also a nontrivial equation on v. Thus, the equation $A=0$ on v is always a nontrivial equation of degree at most 2, and hence we can choose v so that $A\neq 0$. And similarly we get $\lambda_1\neq 0$ for some choice of w. □

5. Collinear case

Lemma 6. *In any deformation of $K(X,Y)$, $|X|,|Y|\geqslant 2$, any two nonadjacent vertices change their mutual distance.*

Proof. Suppose that the distance between two vertices $x_1,x_2\in X$ is fixed under a deformation of $K(X,Y)$. Then, for any $y_i,y_j\in Y$, the shapes of the (possibly degenerate) triangles $x_1x_2y_i$, $x_1x_2y_j$ are fixed under the deformation. Hence the distance between y_i,y_j is fixed. Similarly, the distances among the vertices X are all fixed. This contradicts the definition of a deformation. □

Proposition 4. *Suppose that* $|X| \geqslant 3, |Y| \geqslant 5$, *and* $K(X,Y)$ *admits a deformation. Then* X *lies on a line* ℓ *and* Y *lies on a line perpendicular to* ℓ.

Proof. If all points in $X \cup Y$ are in general position, then $X \cup Y$ lies on an ellipse or a hyperbola or a parabola by Proposition 1. However, by Propositions 2, 3, such cases are impossible. By the same reason, the vertices of $K(X,Y)$ cannot move into a general position by a deformation. Hence there are three vertices that remain collinear during a deformation. Let p_1, p_2, p_3 be such three vertices. Then, these three vertices must belong to the same partite set, for otherwise, two nonadjacent vertices are kept at the same distance, contradicting Lemma 6. So, we may assume that they belong to X.

Now, by superposing a suitable smooth rigid motion, we may suppose that p_1, p_2, p_3 are moving on the x-axis. Further, we may put $p_1 = (0,0)$ (fixed), $p_2 = (\alpha, 0)$ and $p_3 = (\beta, 0)$, where α, β are functions of time t. Let $q = (x, y) = (x(t), y(t))$ be the position of a vertex in Y. The vertex q may cross the x-axis, but it cannot move on the x-axis by Lemma 6. Hence we may assume $y \neq 0$. Since the nontrivial vector field

$$(0,0) \mapsto (0,0), \quad (\alpha,0) \mapsto (\dot{\alpha},0), \quad (\beta,0) \mapsto (\dot{\beta},0)$$

(where $\dot{\alpha}, \dot{\beta}$ denote the derivatives by t) can be extended to an infinitesimal motion of $K(X,Y)$, it follows from Lemma 4 that

$$0 = \begin{vmatrix} x & y & 0 \\ x-\alpha & y & \dot{\alpha}(x-\alpha) \\ x-\beta & y & \dot{\beta}(x-\beta) \end{vmatrix}$$

$$= -y((\beta\dot{\alpha} - \alpha\dot{\beta})x + \alpha\beta(\dot{\beta} - \dot{\alpha})).$$

By Lemma 1, we may suppose that $\dot{\alpha} - \dot{\beta} \neq 0$. Since $y \neq 0$ and $\alpha\beta(\dot{\alpha} - \dot{\beta}) \neq 0$, we must have $\beta\dot{\alpha} - \alpha\dot{\beta} \neq 0$, and

$$x = \frac{\alpha\beta(\dot{\alpha} - \dot{\beta})}{\beta\dot{\alpha} - \alpha\dot{\beta}}.$$

Thus, all vertices of Y lie on a line perpendicular to the x-axis, and hence all vertices of X lie on the x-axis. □

6. The Bottema linkage

The following result is due to Bottema (see Wunderlich [4]):

Theorem 3. *There is a flexible representation* $K(X,Y)$ *of* $K(4,4)$ *in the plane such that the convex hulls of* X *and* Y *are both rectangles.*

Proof. Consider the equation on x, y, z containing the parameter t:

$$(x-t)^2 + (y-z)^2 = a(x-l)^2 + (y+z)^2 = b$$

$$(x-t)^2 + (y-z)^2 = c(x+t)^2 + (y+z)^2 = d$$

where a, b, c, d are positive constants such that $a+d=b+c$. We can choose a, b, c, d suitably so that the above equation has real solutions for some range of t. For example, letting $a=4$, $b=6$, $c=8$ and $d=10$, we have a solution

$$x(t)=\frac{1}{t},\ y(t)=\frac{\sqrt{8-f(t)}\pm\sqrt{6-f(t)}}{2},\ z(t)=\frac{1}{2y(t)},$$

where $f(t)=(t^4+1)/t^2$, which take real values for $\sqrt{2}-1\leqslant t\leqslant\sqrt{2}+1$. Let

$$p_1=(t,z(t)),\qquad p_2=(-t,z(t)),\ p_3=-p_1,\ p_4=-p_2,$$
$$q_1=(x(t),y(t)),\ q_2(-x(t),y(t)),\ q_3=-q_1,\ \ q_4=-q_2,$$

and $X=\{p_1,p_2,p_3,p_4\},Y=\{q_1,q_2,q_3,q_4\}$. Then, varying t from $\sqrt{2}-1$ to $\sqrt{2}+1$, we have a continuous deformation of $K(X,Y)$. □.

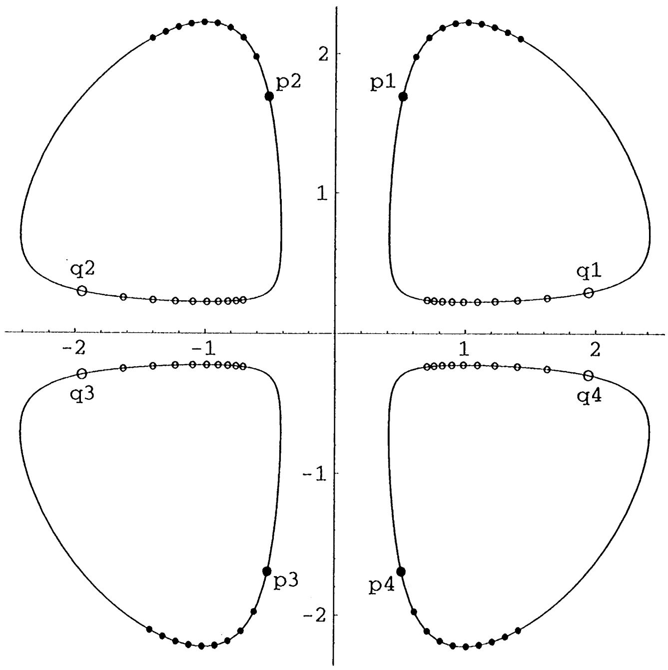

Fig. 1.

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